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A vector bundle proof of Poncelet's closure theorem

Jean Vallès*

Université de Pau et des Pays de l'Adour, Laboratoire de Mathématique- Bât. IPRA, Avenue de l'Université, 64000 PAU, France

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Abstract

There are few different proofs of the celebrated Poncelet closure theorem about polygons simultaneously inscribed in a smooth conic and circumscribed around another. We propose a new proof, based on the link between Schwarzenberger bundles and Poncelet curves.

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1. Introduction

In the town of Saratov where he was a prisoner, Poncelet, continuing the work of Chapple on triangles simultaneously inscribed in a circle and circumscribed around another circle, proved the following generalization (see Figs. 1–3).

Theorem. *Let C and D be two smooth conics in $\mathbb{P}^2(\mathbb{C})$ such that there exists one n-gon (polygon with n sides) inscribed in D and circumscribed around C. Then there are infinitely many such n-gons.*

According to Berger [2, p. 256] this theorem is the nicest result about the geometry of conics.[1] Even if it is, there are few proofs of it. To my knowledge there are only three.

* Tel.: +33 0 559407517.
E-mail address: jean.valles@univ-pau.fr.

[1] As suggested by the referee, we could name also, among the nicest theorems about conics, the one, due to Chasles, which says that there are 3264 conics tangent to five given conics.

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The first proof, published in 1822 and based on infinitesimal deformations, is due to Poncelet [6, p. 362]. Later, Jacobi proposed a new proof based on finite order points on elliptic curves; his proof, certainly the most famous, is explained in a modern way and in detail by Griffiths and Harris (see [4]). In 1870 Weyr proved a Poncelet theorem in space (more precisely for two quadrics) that implies the one above when one quadric is a cone; this proof is explained by Barth and Bauer [1, Theorem 1.1].

Our aim in this short note is to involve vector bundle techniques to propose a new proof of this celebrated result. Poncelet did not appreciate Jacobi's for the reason that it was too far from the geometric intuition. I guess that he would not appreciate our proof either for the same reason.

2. Preliminaries

In all this text the ground field is $\mathbb{C}$. A n-gon consists of the data of n distinct ordered points $a_1, \ldots, a_n$ and n lines $(a_1a_2), \ldots, (a_{n-1}a_n), (a_na_1)$. A complete n-gon consists of the data of n distinct lines in linear general position and their $\binom{n}{2}$ points of intersection, also called vertices.

We say that a n-gon (respectively a complete n-gon) is inscribed in a given curve if this curve passes through its n points (respectively its $\binom{n}{2}$ vertices). We say that a n-gon, or a complete n-gon, is circumscribed around a smooth conic C if the sides of the polygon, i.e. the n lines, are tangent to C.

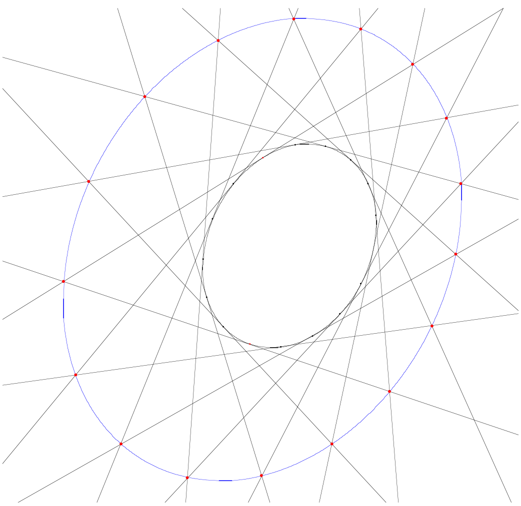

Fig. 1. A 17-gon simultaneously inscribed in a conic and circumscribed around another.

3. Schwarzenberger bundles

First of all let us introduce a vector bundle $E_{n,C}$ naturally associated with any set of n lines tangent to a fixed smooth conic $C \subset \mathbb{P}^2$ (they were defined by Schwarzenberger in [7]). A set of n lines tangent to C corresponds by projective duality to a degree n divisor on the dual conic $C^\vee \subset \mathbb{P}^{2\vee}$. According to the isomorphism $C^\vee \simeq \mathbb{P}^1$ we can consider the subvariety $X \subset \mathbb{P}^2 \times \mathbb{P}^1$ defined by the equation $x_0u^2 + x_1uv + x_2v^2 = 0$ ((x_0, x_1, x_2) are the homogeneous coordinates on $\mathbb{P}^2$ and (u, v) the homogeneous coordinates on $\mathbb{P}^1$) and the projections p and q respectively on $\mathbb{P}^2$ and $\mathbb{P}^1$.

$$\begin{array}{ccc} X & \xrightarrow{q} & \mathbb{P}^1 \\ {\scriptstyle p}\downarrow & & \\ \mathbb{P}^2 & & \end{array}$$

The variety X is the incidence variety of couples (x, l) such that $x \in \mathbb{P}^2$, l is tangent to C and $x \in l$. It is a double cover of $\mathbb{P}^2$ ramified along the curve C defined by the vanishing of the discriminant $x_1^2 - 4x_0x_2 = 0$ of the equation of X in $\mathbb{P}^2 \times \mathbb{P}^1$. If $z \in \mathbb{P}^1$ then $p(q^{-1}(z))$ is a line in $\mathbb{P}^2$ tangent to C. If, instead of considering a point, we are considering a line bundle on $\mathbb{P}^1$ we will find a vector bundle of rank two on $\mathbb{P}^2$ by taking the direct image of its inverse image. Moreover, following Schwarzenberger, we know a very explicit resolution of this bundle. Indeed, tensoring the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(-1, -2) \longrightarrow \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

by $q^*\mathcal{O}_{\mathbb{P}^1}(n)$ and taking its direct image $E_{n,C} := p_*(q^*\mathcal{O}_{\mathbb{P}^1}(n))$ by p we have:

$$0 \longrightarrow \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n-2)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{M} \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n)) \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow E_{n,C} \longrightarrow 0.$$

The map M can be represented by the matrix of linear forms:

$$M = \begin{pmatrix} x_0 & & & \\ x_1 & x_0 & & \\ x_2 & x_1 & \ddots & \\ & x_2 & \ddots & x_0 \\ & & \ddots & x_1 \\ & & & x_2 \end{pmatrix}.$$

Let us show first that a non-zero section $s \in \mathrm{H}^0(E_{n,C})$ corresponds[2] to an n-gon tangent to C. Indeed, since $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n)) = \mathrm{H}^0(E_{n,C})$, the section s corresponds to a divisor D_n of degree n on $\mathbb{P}^1$. For s general enough this divisor consists of n distinct points on $\mathbb{P}^1 \simeq C^\vee$ i.e. of n distinct lines in $\mathbb{P}^2$ tangent to C.

[2] We assume that s is general enough to make sure that the n lines are distinct. Of course it can be proved for any section by extending the definition of n-gon.

Let us now show that the zero locus $Z(s)$ of this section is the set of the $\binom{n}{2}$ vertices of the corresponding complete n-gon. Since $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n)) = \mathrm{H}^0(E_{n,C})$, the section s corresponds also to a hyperplane $H_s \subset \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n)))$. This hyperplane meets the image $v_n(\mathbb{P}^1)$ of $\mathbb{P}^1 \simeq C^\vee$ in $\mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(n)))$ (by the Veronese imbedding v_n) along n points which correspond to the points of the divisor D_n. The section s induces a rational map $\pi_s : \mathbb{P}^2 \longrightarrow \mathbb{P}((E_{n,C})^\vee)$ which is not defined over the zero-scheme $Z(s)$. More precisely let x be a point in $\mathbb{P}^2$ and $L_x \subset \mathbb{P}^{2\vee}$ its dual line. This dual line corresponds by the Veronese morphism to a two-secant line of $v_n(\mathbb{P}^1)$ (call it L_x again). If L_x is not a two-secant line to D_n there is exactly one intersection point $L_x \cap H_s$ which is the image of x by π_s. Conversely the map π_s is not well defined when $L_x \subset H_s$, i.e. when L_x is a two-secant line to D_n, or equivalently when x is a vertex of two tangent lines to C along D_n.

These two facts will be crucial in the forthcoming proofs.

4. Darboux theorem

We can now prove the so-called Darboux theorem [3, p. 248].

Theorem 4.1. *Let $S \subset \mathbb{P}^2$ be a curve of degree $(n-1)$. If there is a complete n-gon tangent to a smooth conic C and inscribed into S, then there are infinitely many of them.*

Remark 4.2. Since the vector space of curves of degree $n-2$ has dimension $\binom{n}{2}$, the degree of a curve passing through the $\binom{n}{2}$ vertices of a complete n-gon is at least equal to $n-1$.

Proof. I recall here a proof already written in [8]. A complete n-gon circumscribed around C and inscribed into S corresponds to a non-zero global section $s \in \mathrm{H}^0(E_{n,C})$ vanishing along its vertices $Z(s)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow E_{n,C} \longrightarrow \mathcal{I}_{Z(s)}(n-1) \longrightarrow 0.$$

By the remark of the previous Section 2, the set $Z(s)$ belongs to S. It implies that the curve S corresponds to a global section of $\mathcal{I}_{Z(s)}(n-1)$. Since the map

$$\mathrm{H}^0(E_{n,C}) \longrightarrow \mathrm{H}^0(\mathcal{I}_{Z(s)}(n-1))$$

is surjective, this section comes from a non-zero section $t \in \mathrm{H}^0(E_{n,C})$ (i.e. another n-gon) such that $Z(t)$ belongs to S. We deduce then that the determinant

$$\mathcal{O}_{\mathbb{P}^2}^2 \xrightarrow{(s,t)} E_{n,C}$$

is the equation of S. In other terms, a general section in the pencil generated by s and t (i.e. a general linear combination of s and t) corresponds to a complete n-gon circumscribed around C and inscribed in S. This proves the theorem. □

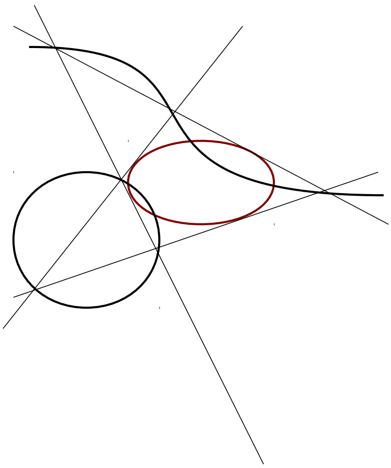

Fig. 2. A complete circumscribed 4-gon and a cubic Poncelet curve.

These curves described by Darboux are called *Poncelet curves*. When $n = 5$ they are the so-called Luröth quartics (see [5]).

5. Poncelet theorem

Let us now consider n-gons that are simultaneously inscribed in a smooth conic and circumscribed around another one. For these configurations Poncelet obtained his famous result [6, p. 362]. We prove it now, with the help of Poncelet curves defined above.

Theorem 5.1. *Let $C \subset \mathbb{P}^2$ and $D \subset \mathbb{P}^2$ be two smooth conics such that there exists one n-gon inscribed in D and circumscribed around C. Then there are infinitely many of such n-gons.*

Proof. Let us consider one such n-gon and let us denote its sides by $l_1, \ldots, l_n$. By hypothesis, there is a non-zero section $s \in \mathrm{H}^0(E_{n,C})$ vanishing along the $\binom{n}{2}$ vertices $l_i \cap l_j$ (for $1 \leq i, j \leq n$ and $i \neq j$) of these lines. We denote by $Z(s)$ the set of these vertices.

Let us tensor the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow E_{n,C} \longrightarrow \mathcal{I}_{Z(s)}(n-1) \longrightarrow 0$$

by $\mathcal{O}_D$. Since $D \cap Z(s)$ consists of n-points, it induces the following decomposition of $E_{n,C}$ along D:

$$E_{n,C} \otimes \mathcal{O}_D = \mathcal{O}_D\left(\frac{n-2}{2}\right) \oplus \mathcal{O}_D\left(\frac{n}{2}\right).$$

According to this decomposition, we consider the following exact sequence:

$$0 \longrightarrow F \longrightarrow E_{n,C} \longrightarrow \mathcal{O}_D\left(\frac{n-2}{2}\right) \longrightarrow 0$$

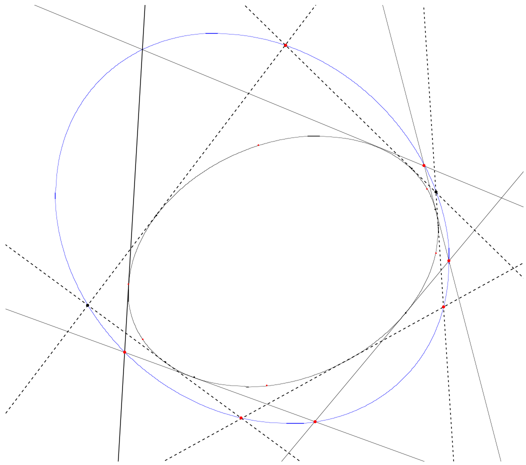

Fig. 3. Pentagons inscribed and circumscribed.

where F is a rank two vector bundle over $\mathbb{P}^2$. Taking the cohomology long exact sequence we verify immediately that $h^0(F) \geq 2$. Then, let us consider a pencil of sections of F and also the pencil of sections of $E_{n,C}$ induced by it. We obtain a commutative diagram:

$$\begin{array}{ccccccccc}
 & & \mathcal{O}_{\mathbb{P}^2}^2 & = & \mathcal{O}_{\mathbb{P}^2}^2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F & \longrightarrow & E_{n,C} & \longrightarrow & \mathcal{O}_D\left(\frac{n-2}{2}\right) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{L}_1 & \longrightarrow & \mathcal{L}_2 & \longrightarrow & \mathcal{O}_D\left(\frac{n-2}{2}\right) & \longrightarrow & 0.
\end{array}$$

The sheaf $\mathcal{L}_2$ is supported by a curve Γ_2 of degree $(n-1)$ that is the determinant of a pencil of sections of $E_{n,C}$. This curve Γ_2 is a Poncelet curve. Then a general point on Γ_2 is a vertex of a complete n-gon inscribed in Γ_2 and circumscribed around C. Moreover any intersection point of the n lines forming the n-gon with Γ_2 is a vertex of this n-gon (this is clear by the Bézout theorem since $n(n-1) = 2 \times \binom{n}{2}$). Let Γ_1 be the curve supporting the sheaf $\mathcal{L}_1$.[3] We have of course $\Gamma_2 = D \cup \Gamma_1$. Then D is an irreducible component of a Poncelet curve and by the way any general point on D is a vertex of the complete n-gon inscribed in Γ_2. Then this configuration meets the conic D in at least (because there are n lines) and at most (because they are vertices and the decomposition of the bundle along D is fixed) n points, so exactly n points, each counting twice. □

[3] As the referee pointed out to me, the existence of this curve can be seen directly, by computing the dimension of the vector space of polynomials of degree $(n-3)$ passing through the $\binom{n}{2} - n$ vertices that do not belong to D. This remark leads to a simplified version of this proof. The referee also reminded me that, according to Darboux ([3], livre III, chapitre II page 252), this curve is a union of $\frac{n-3}{2}$ conics for n odd, or a union of $\frac{n-4}{2}$ conics and a line for n even.

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