On Jacobi–Bertrand's proof of a Theorem of Poncelet

by

I. J. SCHOENBERG (West Point)

1. Introduction

A result from the golden age of Geometry is the following:

**Theorem 1 (Poncelet).** Let $\mathcal{C}$ and $\mathcal{C}'$ be two ellipses in the plane, $\mathcal{C}'$ being interior to $\mathcal{C}$. From a point $P$ on $\mathcal{C}$ we draw a tangent to $\mathcal{C}'$ intersecting $\mathcal{C}$ again in $P_1$. Again, from $P_1$ we draw the tangent to $\mathcal{C}'$, intersecting $\mathcal{C}$ at $P_2$, and we continue this construction $n$ times until we obtain the tangent $P_{n-1}P_n$ to $\mathcal{C}'$, with $P_n \in \mathcal{C}$. The result is the polygonal line $PP_1 \ldots P_n$ inscribed in $\mathcal{C}$ and circumscribed to $\mathcal{C}'$. If after one revolution along $\mathcal{C}$ we have that $P_n = P$, so that the polygon $PP_1 \ldots P_n$ is closed, then this will happen no matter what point $P$ on $\mathcal{C}$ we start from.

We also state the following two special cases of Theorem 1.

**Theorem 2.** Theorem 1 holds for the special case when $\mathcal{C}$ and $\mathcal{C}'$ are circles.

**Theorem 3.** Theorem 1 holds if $\mathcal{C}$ is an ellipse and $\mathcal{C}'$ is a circle having its center $O$ in the center of $\mathcal{C}$.

JACOBI gave a beautiful proof of Theorem 2 which was improved by J. BERTRAND in [1, pp. 575–577]. BERTRAND liked it so much that he reproduced it also at the end of his note on elliptic functions in [2, pp. 534–536].

Here we want to show 1 (by means of suitable projective maps of the plane, a proof of Theorem 1 may be reduced to a proof of Theorem 3. 2) a simplified version of JACOBI's idea leads to a direct proof of Theorem 3.

2. Reducing Theorem 1 to Theorem 3

Let $\mathcal{D}$ and $\mathcal{D}'$ denote the closed elliptical domains bounded by $\mathcal{C}$ and $\mathcal{C}'$, respectively. Let $P \in \mathcal{D}$, and let $\Lambda$ denote the polar line of $P$ with respect to the ellipse $\mathcal{C}$. As $\Lambda$ does not intersect the interior of $\mathcal{D}$, it follows that $\Lambda \cap D' = \emptyset$. Therefore the pole $P'$ of $\Lambda$ with respect to the ellipse $\mathcal{C}'$ is in the interior of $D'$: $P' \in D'$. This construction
produces a mapping

\[ T : D \rightarrow D' \subseteq D . \]

By Brouwer's fixed-point theorem there is a point \( P_0 \) such that \( P_0' = TP_0 = P_0 \). Let \( \Delta_0 \) be the polar line of \( P_0 \) with respect to \( C \), as well as with respect to \( C' \). Now we perform a projective map changing \( \Delta_0 \) into the line at infinity, while \( P_0 \) is mapped into the point \( O \). If we denote by \( C_1 \) and \( C'_1 \) the images of \( C \) and \( C' \), it follows that \( O \) must be the common center of the ellipses \( C_1 \) and \( C'_1 \).

Finally, an obvious affine map that leaves the common center \( O \) in place, will change \( C'_1 \) into a circle \( C'_2 \), and \( C_1 \) into an ellipse \( C_2 \).

The final result is that the original ellipses \( C, C' \), have been projectively mapped into the ellipse \( C_2 \) and the circle \( C'_2 \), respectively, the last two curves having the common center \( O \). Now it is clear that Theorem 3 implies Theorem 1.

3. A proof of Theorem 3 using the Jacobi–Bertrand's idea

Let

(1) \[ C : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (b < a) \]

and

(2) \[ C' : x^2 + y^2 = r^2 \quad (r < b) . \]

Furthermore let

(3) \[ P = (a \cos \phi, b \sin \phi), \quad P' = (a \cos \phi, a \sin \phi) \]

be a point of \( C \) and its corresponding point on the principal circle \( \Gamma \) of \( C \).

We draw the tangent \( PP_1 \) touching \( C' \) at \( M \), and let

(4) \[ \not AOP' = \phi, \quad \not AOP_1 = \phi_1 , \]

our objective being to find the value of the derivative

(5) \[ \frac{d\phi_1}{d\phi} . \]

To find it we consider the affine map

(6) \[ T : x = x', \quad y = \frac{b}{a} y' , \]
which maps $\Gamma$ into $C$. Let $M' = T^{-1}M$. If we let $P$ move along $C$, the line $PMP_1$ being tangent to $C$ at $M$, then the line $P'M'P_1$ will envelope an ellipse $T^{-1}C'$, and will be tangent to it at $M'$. However, the line $P'M'P_1$ intersects the circle $\Gamma$ in equal angles, and it follows that

\[
\frac{d\varphi_1}{d\varphi} = \frac{M'P_1}{M'P'} = \frac{MP_1}{MP}.
\]

We also find that

\[
(MP)^2 = (OP)^2 - (OM)^2 = a^2 \cos^2 \varphi + b^2 \sin^2 \varphi - r^2 =
\]

\[
= a^2(1 - \sin^2 \varphi) + b^2 \sin^2 \varphi - r^2 = a^2 - r^2 - (a^2 - b^2) \sin^2 \varphi
\]

and finally that

\[
MP = \sqrt{a^2 - r^2} \cdot \sqrt{1 - k^2 \sin^2 \varphi}
\]

where

\[
k^2 = \frac{a^2 - b^2}{a^2 - r^2}.
\]

Similarly

\[
MP_1 = \sqrt{a^2 - r^2} \cdot \sqrt{1 - k^2 \sin^2 \varphi_1}.
\]
For the derivative (5) we now find, by (7), (8), and (10), the expression

\[ \frac{d\varphi_1}{d\varphi} = \frac{1}{\sqrt{1 - k^2 \sin^2 \varphi}} \frac{d\varphi_1}{\sqrt{1 - k^2 \sin^2 \varphi}} \]

whence the differential equation

\[ \frac{d\varphi_1}{\sqrt{1 - k^2 \sin^2 \varphi_1}} = \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \]  

As in Jacobi's proof of Theorem 2, we consider the integral

\[ J(\varphi, \varphi_1) = \int_{\varphi_1}^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \]

whose differential

\[ dJ(\varphi, \varphi_1) = \frac{d\varphi_1}{\sqrt{1 - k^2 \sin^2 \varphi_1}} - \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \]

vanishes, in view of (11). It follows that

\[ J(\varphi, \varphi_1) = \text{a constant } \omega, \text{ independent of } \varphi . \]

If we now perform the successive constructions of the tangents \( n \) times, obtaining \( P_1P_2, \ldots, P_{n-1}P_n \), all tangent to \( C' \), and the angles

\[ \angle AOP_2 = \phi_2, \ldots, \angle AOP_n = \phi_n, \]

then, by (13), we find that

\[ \int_{\phi_1}^{\phi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = J(\phi, \phi_1) + J(\phi_1, \phi_2) + \ldots + J(\phi_{n-1}, \phi_n) = n\omega . \]

In the case that the polygon \( PP_1 \ldots P_n \) closes after one revolution along \( C \), hence \( P_n = P \), then also \( P_n = P' \), and therefore \( \phi_n = \phi + 2\pi \). But then the equation (14) becomes

\[ \int_{\phi}^{\phi + 2\pi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = n\omega . \]
This equation remains valid no matter how we choose $\phi$, because $2\pi$ is a period of the integrand of (15). This establishes Theorem 3.

If the polygon $PP_1 \ldots P_n$ should close only after $N$ revolutions along the ellipse $C$, then again (15) must hold, provided that we replace the upper limit of the integral by $\phi + 2\pi N$. Since $2\pi N$ is again a period of the integrand, Theorem 3 holds also in this case.

References