Studies in Pure Mathematics
To the Memory of Paul Turán

On Jacobi–Bertrand's proof of a Theorem of Poncelet

by

I. J. SCHOENBERG (West Point)

1. Introduction

A result from the golden age of Geometry is the following:

Theorem 1 (Poncelet). *Let C and C' be two ellipses in the plane, C' being interior to C. From a point P on C we draw a tangent to C' intersecting C again in P_1. Again, from P_1 we draw the tangent to C', intersecting C at P_2, and we continue this construction n times until we obtain the tangent $P_{n-1}P_n$ to C', with $P_n \in C$. The result is the polygonal line $PP_1 \ldots P_n$ inscribed in C and circumscribed to C'. If after one revolution along C we have that $P_n = P$, so that the polygon $PP_1 \ldots P_n$ is closed, then this will happen no matter what point P on C we start from.*

We also state the following two special cases of Theorem 1.

Theorem 2. *Theorem 1 holds for the special case when C and C' are circles.*

Theorem 3. *Theorem 1 holds if C is an ellipse and C' is a circle having its center O in the center of C.*

Jacobi gave a beautiful proof of Theorem 2 which was improved by J. Bertrand in [1, pp. 575–577]. Bertrand liked it so much that he reproduced it also at the end of his note on elliptic functions in [2, pp. 534–536].

Here we want to show 1 (by means of suitable projective maps of the plane, a proof of Theorem 1 may be reduced to a proof of Theorem 3. 2) a simplified version of Jacobi's idea leads to a direct proof of Theorem 3.

2. Reducing Theorem 1 to Theorem 3

Let D and D' denote the closed elliptical domains bounded by C and C', respectively. Let $P \in D$, and let Δ denote the polar line of P with respect to the ellipse C. As Δ does not intersect the interior of D, it follows that $\Delta \cap D' = \emptyset$. Therefore the pole P' of Δ with respect to the ellipse C' is in the interior of D': $P' \in D'$. This construction produces a mapping

$$T: D \to D' \subset D.$$

By BROUWER's fixed-point theorem there is a point P_0 such that $P'_0 = TP_0 = P_0$. Let Δ_0 be the polar line of P_0 with respect to C, as well as with respect to C'. Now we perform a projective map changing Δ_0 into the line at infinity, while P_0 is mapped into the point O. If we denote by C_1 and C'_1 the images of C and C', it follows that O must be the common center of the ellipses C_1 and C'_1.

Finally, an obvious affine map that leaves the common center O in place, will change C'_1 into a circle C'_2, and C_1 into an ellipse C_2.

The final result is that the original ellipses C, C', have been projectively mapped into the ellipse C_2 and the circle C'_2, respectively, the last two curves having the common center 0. Now it is clear that Theorem 3 implies Theorem 1.

3. A proof of Theorem 3 using the Jacobi–Bertrand's idea

Let

(1) $$C: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (b < a)$$

and

(2) $$C': x^2 + y^2 = r^2 \quad (r < b).$$

Furthermore let

(3) $$P = (a \cos \varphi, b \sin \varphi), \quad P' = (a \cos \varphi, a \sin \varphi)$$

be a point of C and its corresponding point on the principal circle Γ of C.

We draw the tangent PP_1 touching C' at M, and let

(4) $$\sphericalangle AOP' = \varphi, \quad \sphericalangle AOP'_1 = \varphi_1,$$

our objective being to find the value of the derivative

(5) $$\frac{d\varphi_1}{d\varphi}.$$

To find it we consider the affine map

(6) $$T: x = x', \quad y = \frac{b}{a} y',$$

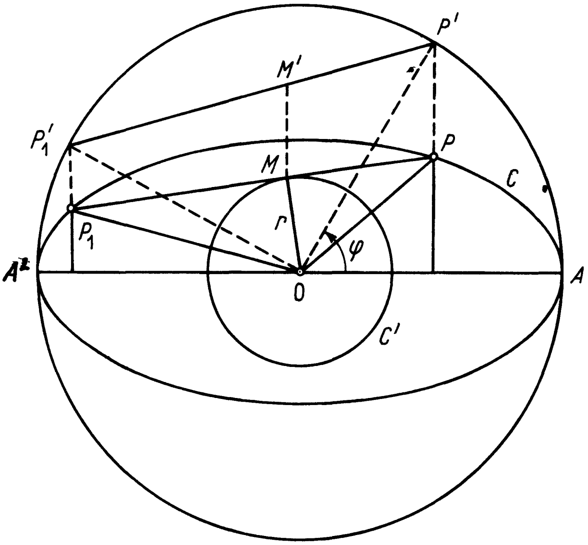

Fig. 1

which maps Γ into C. Let $M' = T^{-1}M$. If we let P move along C, the line PMP_1 being tangent to C at M, then the line $P'M'P'_1$ will envelope an ellipse $T^{-1}C'$, and will be tangent to it at M'. However, the line $P'M'P'_1$ intersects the circle Γ in equal angles, and it follows that

$$\frac{d\varphi_1}{d\varphi} = \frac{M'P'_1}{M'P'} = \frac{MP_1}{MP}. \tag{7}$$

We also find that

$$(MP)^2 = (OP)^2 - (OM)^2 = a^2 \cos^2 \varphi + b^2 \sin^2 \varphi - r^2 =$$

$$= a^2(1 - \sin^2 \varphi) + b^2 \sin^2 \varphi - r^2 = a^2 - r^2 - (a^2 - b^2) \sin^2 \varphi$$

and finally that

$$MP = \sqrt{a^2 - r^2} \cdot \sqrt{1 - k^2 \sin^2 \varphi} \tag{8}$$

where

$$k^2 = \frac{a^2 - b^2}{a^2 - r^2}. \tag{9}$$

Similarly

$$MP_1 = \sqrt{a^2 - r^2} \cdot \sqrt{1 - k^2 \sin^2 \varphi_1}. \tag{10}$$

For the derivative (5) we now find, by (7), (8), and (10), the expression

$$\frac{d\varphi_1}{d\varphi} = \frac{\sqrt{1-k^2 \sin^2 \varphi_1}}{\sqrt{1-k^2 \sin^2 \varphi}},$$

whence the differential equation

$$\frac{d\varphi_1}{\sqrt{1-k^2 \sin^2 \varphi_1}} = \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}. \tag{11}$$

As in Jacobi's proof of Theorem 2, we consider the integral

$$J(\varphi, \varphi_1) = \int_{\varphi}^{\varphi_1} \frac{d\vartheta}{\sqrt{1-k^2 \sin^2 \vartheta}}, \tag{12}$$

whose differential

$$dJ(\varphi, \varphi_1) = \frac{d\varphi_1}{\sqrt{1-k^2 \sin^2 \varphi_1}} - \frac{d\varphi}{\sqrt{1-k^2 \sin^2 \varphi}}$$

vanishes, in view of (11). It follows that

$$J(\varphi, \varphi_1) = \textit{a constant } \omega, \textit{ independent of } \varphi. \tag{13}$$

If we now perform the successive constructions of the tangents n times, obtaining $P_1P_2, \ldots, P_{n-1}P_n$, all tangent to C', and the angles

$$\measuredangle AOP'_2 = \varphi_2, \ldots \measuredangle AOP'_n = \varphi_n,$$

then, by (13), we find that

$$\int_{\varphi}^{\varphi_n} \frac{d\vartheta}{\sqrt{1-k^2 \sin^2 \vartheta}} = J(\varphi, \varphi_1) + J(\varphi_1, \varphi_2) + \ldots + J(\varphi_{n-1}, \varphi_n) = n\omega. \tag{14}$$

In the case that the polygon $PP_1 \ldots P_n$ closes after one revolution along C, hence $P_n = P$, then also $P'_n = P'$, and therefore $\varphi_n = \varphi + 2\pi$. But then the equation (14) becomes

$$\int_{\varphi}^{\varphi+2\pi} \frac{d\vartheta}{\sqrt{1-k^2 \sin^2 \vartheta}} = n\omega. \tag{15}$$

This equation remains valid no matter how we choose φ, because 2π *is a period of the integrand of* (15). This establishes Theorem 3.

If the polygon $PP_1 \ldots P_n$ should close only after N revolutions along the ellipse C, then again (15) must hold, provided that we replace the upper limit of the integral by $\varphi + 2\pi N$. Since $2\pi N$ is again a period of the integrand, Theorem 3 holds also in this case.

References

[1] J. BERTRAND, *Traité de Calcul Différentiel et de Calcul Intégral*, vol. 2, Gauthier–Villars, Paris, 1870.
[2] J. BERTRAND, *Note sur la Théorie des Fonctions Élliptiques.* An Appendix in J. M. C. DUHAMEL, *Éléments de Calcul Infinitésimal*, vol. 2, pp. 526–536, Gauthier–Villars, Paris, 1876.

DEPARTMENT OF MATHEMATICS
UNITED STATES MILITARY ACADEMY
WEST POINT, NEW YORK 10996