but the series has been extended by me (1947) as follows:

\[
n! \left[ \frac{1}{n} \left( 1 - \frac{1}{2n^3} + \frac{1}{3n^4} + \frac{40}{21n^6} + \frac{991}{120n^8} + \frac{105301}{720n^{10}} + \frac{4386229}{5040n^{12}} - \frac{16384008239}{3628800n^{14}} - \frac{618211198039}{3628800n^{16}} - \ldots \right) \right].
\]

It is instructive to compare this last result with (7). The two asymptotic expansions are identical as far as the fourth term, but differ from the fifth term onwards. This throws interesting light on some conjectures of Erdős and Kacskásky about the asymptotic number of the general \( n \) by \( k \) Latin rectangle. I hope to examine the implications in a subsequent paper.

PRESIDENTS COLLEGE,
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References

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PONCELET PORISM IN TWO CIRCLES

If the chord joining the points $A(R\cos \theta, R \sin \theta)$ and $A_1(R \cos \theta_1, R \sin \theta_1)$ on the circle $B \equiv x^2 + y^2 - R^2 = 0$ touches the circle $s \equiv (x - d)^2 + y^2 - r^2 = 0$, we must have

$$R \cos \theta - \theta_1 - d \cos \theta \theta_1 + \theta_1 = \pm r,$$

the sign to be taken positive if $s$ be the in-circle, negative otherwise. In what follows, we shall choose the positive sign throughout and write $-r$ for $r$ or $-c$ for $c$ to derive $C_i(n)$ from $C_i(n)$ or vice versa. On writing $t_i = \tan \theta_i$, the condition of contact may be rewritten as

$$u^2(t_1 + t_{11})^2 + v^2(1 - (t_1 + t_{11})^2) = w^2(1 + (t_1 + t_{11})^2).$$

(1.1)

This is the quadratic relation connecting the points $A_0, A_1, \ldots, A_n$. Assuming the poristic nature of the problem, no loss of generality occurs by setting $\theta_0 = 0$, giving $t_0 = 0$ and $t_1^2 = w^2 - v^2$. Choosing the positive sign for $t_1$, we may derive two values for $t_1$, one of which will be $t_1$. Repeating the latter value, we get a unique value for $t_1$. If now the value of $t_1, t_1, t_1, \ldots, t_1$ be used, the unique quadratic relation will determine two values for $t_1, t_2, \ldots, t_n$ for which the in-circle, $t_1, t_2, \ldots, t_n$ is uniquely determined. This establishes the induction, leading to unique values for each of $t_1, t_2, \ldots, t_n$ for all integral values of $n$.

The condition that $A_0, A_1$ should touch $s$ is represented by

$$u^2(t_1 + t_{11})^2 + v^2(1 - (t_1 + t_{11})^2) = w^2(1 + (t_1 + t_{11})^2).$$

(1.2)

From (1.1) and (1.2) it follows that

$$t_1 + t_{11} = \frac{-u^2 + v^2 + w^2}{w^2}, \quad 2l_1$$

(1.31)

and

$$t_1 + t_{11} = \frac{t_1^2 + t_1}{1 - l_1 t_1^2},$$

(1.32)

so that

$$t_1 + t_{11} = \frac{-u^2 + v^2 + w^2}{w^2}, \quad 2l_1$$

(1.33)

If we now eliminate $t_1$ between (1.1) and (1.2), we obtain

$$a^2(t_1 + t_{11})^2 + b^2(1 - (t_1 + t_{11})^2) = c^2(1 + (t_1 + t_{11})^2).$$

(1.4)

From the similarity of (1.1) and (1.4), it follows that replacing $a, b, c$ by $u, v, w$ in the three equations (1.3) would lead to

$$t_1 + t_{11} = \frac{-u^2 + v^2 + w^2}{w^2}, \quad 2l_1$$

(1.51)

and

$$l_1 + l_{11} = \frac{t_1^2 + t_1}{1 - l_1 t_1^2},$$

(1.52)

$$t_1 + t_{11} = \frac{2l_1 - u^2 + v^2 + w^2}{w^2},$$

(1.53)

and

$$l_1 + l_{11} = \frac{-u^2 + v^2 + w^2}{w^2}, \quad 2l_1$$

(1.54)
From a comparison of (1.1) and (1.4), it follows further that to convert \( t_i \) into \( t_r \) any one of the following equivalent transformations would suffice:
\[
\begin{align*}
\alpha^2 &\to \beta^2, \\
\alpha^2 &\to (-\alpha^2 + \beta^2 + \gamma^2)^{-1}, \\
\alpha^2 &\to \left(\alpha^2 - \beta^2 + \gamma^2\right)^{-1}, \\
\beta^2 &\to \gamma^2, \\
\beta^2 &\to \left(\alpha^2 + \beta^2 + \gamma^2\right)^{-1}
\end{align*}
\]

This is the general duplicating transformation which we denote by \( T_{\alpha} \). If \( T_{\alpha} \), operates on any homogeneous function of \( \alpha, \beta, \gamma \) of degree 4, it is converted into another homogeneous function of \( \beta, \gamma, \alpha \) of degree 4. The form \( \alpha^2 \to \beta^2, \beta^2 \to \gamma^2, \gamma^2 \to \alpha^2 \), however, requires care in operation. If the porism is in its simplest rational form, which is the form preferred by Chaundy, duplication does not always take place at an operation.

Four cases may be distinguished:
\[
\begin{align*}
T_{\alpha} &\equiv \alpha \to \beta, \quad \beta \to \gamma, \quad \gamma \to \alpha \\
T_{\beta} &\equiv \alpha \to -\beta, \quad \beta \to \gamma, \quad \gamma \to \alpha \\
T_{\gamma} &\equiv \alpha \to \beta, \quad \beta \to -\gamma, \quad \gamma \to \alpha \\
T_{\nu} &\equiv \alpha \to \beta, \quad \beta \to \gamma, \quad \gamma \to -\alpha
\end{align*}
\]

In practice, it is readily found that \( T_{\alpha} \) only rationalizes the porism and does nothing else; \( T_{\beta} \) rationalizes, interchanges \( R \) and \( d \) and converts \( r \) into \( -r \); \( T_{\gamma} \) interchanges \( R \) and \( d \); and finally that \( T_{\nu} \) rationalizes and changes \( r \) into \( -r \). These transformations will prove their worth in the subsequent discussion.

Repeating on (1.4) the operation by which (9.4) was derived from (1.1) and by merely using \( T_{\nu} \), we obtain:
\[
\begin{align*}
\left(-\alpha^2 + \beta^2 + \gamma^2\right)^{-1}(t_{\alpha} + t_{\beta} + t_{\gamma})^2 + (\alpha^2 - \beta^2 + \gamma^2)^{-1}(1 - t_{\alpha} + t_{\beta} - t_{\gamma})^2 &= (\alpha^2 - \gamma^2)^{-2} \left(1 + t_{\alpha} + t_{\beta} + t_{\gamma}\right)^2
\end{align*}
\]

from which follows:
\[
\begin{align*}
l_{\alpha} + l_{\beta} &= 2l_{\gamma} \left[-\alpha^2 + \beta^2 + \gamma^2\right] + 10 \beta^2 \left(\alpha^2 + \beta^2 - \gamma^2\right) + 10 \beta^2 \alpha^2
\end{align*}
\]

and
\[
\begin{align*}
l_{\beta} + l_{\gamma} &= \frac{1}{\left(\alpha^2 - \beta^2 + \gamma^2\right)^2} \left(\alpha^2 - \gamma^2\right)^{-2} \left(1 + t_{\alpha} + t_{\beta} + t_{\gamma}\right)^2
\end{align*}
\]

It follows that the transformation which converts \( t_i \) into \( t_r \) in \( T_{\alpha} \), \( T_{\beta} \), \( T_{\gamma} \), or \( T_{\nu} \) equivalent to:
\[
\begin{align*}
t \to (-\alpha^2 + \beta^2 + \gamma^2)^{-1} - 2\beta^2 \gamma^2 + 2\beta^2 \alpha^2 + 2\beta^2 \gamma^2^{-1}
\end{align*}
\]

We next seek to eliminate \( t_{\alpha} \) between:
\[
\begin{align*}
\alpha(t_{\alpha} + t_{\beta} + t_{\gamma})^2 + \beta^2 (1 - t_{\alpha} + t_{\beta} - t_{\gamma})^2 &= \alpha^4 (t_{\alpha} + t_{\beta} + t_{\gamma})^2
\end{align*}
\]

and:
\[
\begin{align*}
\alpha(t_{\alpha} + t_{\beta} + t_{\gamma})^2 + \beta^2 (1 - t_{\alpha} + t_{\beta} - t_{\gamma})^2 &= \alpha^4 (t_{\alpha} + t_{\beta} + t_{\gamma})^2
\end{align*}
\]

reaching (after wading through a mass of algebra) the objective in the form:
\[
\begin{align*}
\alpha^4 (t_{\alpha} + t_{\beta} + t_{\gamma})^2 = \alpha^4 (t_{\alpha} + t_{\beta} + t_{\gamma})^2
\end{align*}
\]

As the first factor would imply a retreating in the path \( A_0 A_1 A_3 \ldots \) we reject it. The required eliminant may be written as:
\[
\begin{align*}
\alpha^4 (t_{\alpha} + t_{\beta} + t_{\gamma})^2 = \alpha^4 (t_{\alpha} + t_{\beta} + t_{\gamma})^2
\end{align*}
\]

It follows that the trisecting transformation \( T_{\gamma} \) converting \( t_i \) into \( t_r \), is given by any one of the following equivalent forms:
\[
\begin{align*}
t_i &\to \left(-\alpha + \beta + \gamma\right)^{-1} \gamma, \\
\gamma &\to \left(-\alpha + \beta + \gamma\right)^{-1} \gamma, \\
\beta &\to \left(-\alpha + \beta + \gamma\right)^{-1} \gamma
\end{align*}
\]

From the involutions (1.8), it follows that:
\[
\begin{align*}
\alpha(t_{\alpha} + t_{\beta} + t_{\gamma})^2 &= \left(-\alpha + \beta + \gamma\right)^{-1} \gamma, \\
\gamma &\to \left(-\alpha + \beta + \gamma\right)^{-1} \gamma, \\
\beta &\to \left(-\alpha + \beta + \gamma\right)^{-1} \gamma
\end{align*}
\]

The transformations \( T_{\alpha} \), \( T_{\beta} \), \( T_{\gamma} \), etc. will be obtained by a different method in the subsequent work.
by operating with $T_2$ on the involution connecting $t_0, t_{1.2}$, we may eliminate $t_{1.2}$ between the two involutions connecting respectively $t_0, t_{1.2}, t_3$, and $t_0, r, t_{1.2}$, and reject the retrogressing factor:

$$a^2(t_1 + t_{1.2}) + b^2(1 - t_1 - t_{1.2}) - c^2(1 + t_1 - t_{1.2})^2$$

Hence is obtained the involution connecting $t_0, t_{1.2}$. The induction is thus completed and we have proved the existence (for all values of $i$) of a quadratic involution of the form

$$a^2(t_1 + t_{1.2}) + b^2(1 - t_1 - t_{1.2}) = c^2(1 + t_1 - t_{1.2})^2$$

(2.1)

It follows that

$$t_{1.2} = t_{1} - t_{2}$$

(2.2)

for all values of $i$ and $r$, provided we agree that $t_0 = -t_1$, and that $u^2 \rightarrow a^2$, $v^2 \rightarrow b^2$, $w^2 \rightarrow c^2$ defines the general replicating transformation $T_2$,

3

From the initial assumption $\theta = 0$, it follows that if an $n$-agon be inscribed in $S$ and circumscribed about $s$, the $n$-agon will be symmetrically about the line joining the centres of the two circles.

If $n = 2m$, then $\theta = \pi$ would be an even multiple of $\pi$ and hence $t_0 = t_1$. Since the polygon in this case is a rectangle, the diagonal $A$ must coincide at a finite point and hence the portion given by $t_0 = 1$ can only be the portion corresponding to the "inseparable" case. Hence $\theta(2m)$ is simply given by $t_0 = 1$.

To obtain the portion of $(2m+1)-gon$, we need only remember that $\theta_n = \theta_{n+1}$, and respectively be of the forms $(2m+1)-a$ and $(2m+1)-c$. It follows that the portion $C_r(2m+1)$ is given by $t_0 = 1$. But from (2.2),

$$t_0 = t_0$$

where the denominator vanishes but the numerator does not as $t_0 = t_0^2$ would imply $\theta$ to be a multiple of $2\pi$, which it is not. As in finite, it follows that $t_0 = 1$ must also lead to the same portion.

We could also deduce the portion from the general condition $t_0 = t_0^2 = 1$. In each individual case, it will have to be decided which of these various conditions would lead to the portion most rapidly. But when $(2m+1)$ is prime, the simplest method is to set $t_0 = t_0^2 = 1$. When $(2m+1)$ is composite, the problem must become troublesome, as irrelevant factors are bound to enter into the portion and have to be removed at the cost of great labour, before the relevant portion is ready for exhibition.

Another point in this connection may also be mentioned. The portion thus obtained relates to the in-circle. The portion $C_r(2m+1)$ corresponding to the ex-circle is rapidly deduced by changing $c$ into $-c$ in $C_r(2m+1)$.

**Poncelet's Porism in Two Circles**

We next come to the portion of a $(4m+2)$-agon for the "separable" case. The relevant condition can be deduced by considering only one half of the polygon, which lies symmetrically with respect to the common diameter. The situation in view can only arise if $a = b = 0$, and consequently the polygon has to be referred to the starting-point, and then to be converted on the other side of the bounding line. This implies that $t_0 = t_1$ is the condition for the portion of a "separable" case. This gives $C_2(4m+2)$. The condition $C_r(4m+2)$ may be deduced from $C_r(2m+1)$ by merely changing the sign of $c$. The transformation $T_2$ can rapidly build up all portions of the type $C_r(2m+1)$. The only case that would not be covered by the argument would be the case $C_r(2)$, but $T_2$ supplies all the relevant porisms if only $C_r(4)$ is known. To obtain $C_r(4)$, it is only necessary to observe that $A$ is one of the common points of $s$ and $S$, and this can only happen if $R = \pm d$. The condition $R = \pm d$ will be known as $C_r(4)$ and the condition $R = \pm d$ as $C_r(8)$, although in this particular case the change of sign of $c$ to convert the one form into the other becomes meaningless.

The various formulae connecting $t_i$ have been used to obtain the values of $t_i$, up to $t_i = 10$. To condense the formulae, the following abbreviations have been used:

$$e_1 = a + b + c$$
$$e_2 = a - b + c$$
$$e_3 = a + b - c$$
$$e_4 = a - b - c$$
$$e_5 = b + c$$
$$e_6 = b - c$$

$$l_i = b_i + e_i + c_i$$
$$m_i = b_i + e_i - c_i$$

$$e_i = b_i + e_i$$

$$P_i = b_i + e_i$$

The various forms for $t_i$ have been used to obtain the values of $t_i$, up to $t_i = 10$. To condense the formulae, the following abbreviations have been used:

$$e_1 = a + b + c$$
$$e_2 = a - b + c$$
$$e_3 = a + b - c$$
$$e_4 = a - b - c$$
$$e_5 = b + c$$
$$e_6 = b - c$$

$$l_i = b_i + e_i + c_i$$
$$m_i = b_i + e_i - c_i$$

$$e_i = b_i + e_i$$

$$P_i = b_i + e_i$$

The various forms for $t_i$ have been used to obtain the values of $t_i$, up to $t_i = 10$. To condense the formulae, the following abbreviations have been used:
The $t$'s may now be written as

\[ t_0 = 0 \]

\[ t_3 = \frac{c-b}{a} t_1 \]

\[ t_3 = \frac{a+b}{a} t_1 \]

\[ t_5 = 2aE_i t_3 \]

\[ t_5 = \frac{F}{P} t_3 \]

\[ t_7 = 4a^2E_i E_j t_9 \]

\[ t_7 = \frac{c+c_i h_j}{c+c_i h_j} t_9 \]

\[ t_9 = \frac{H_i H_j}{H_i H_j} t_9 \]

\[ t_9 = \frac{E_i E_j G_i G_j K_1}{E_i E_j G_i G_j K_1} t_9 \]

The isomorphisms obtained by methods sketched above must all occur in their rationalized forms. To obtain Chebyshev's irrational forms, we must apply the inverse transformation of $T_9$ and denote it by $T_9^{-1}$. This obviously implies

\[ a \rightarrow u, \quad b \rightarrow v, \quad c \rightarrow w \]

It would accordingly be worth while to work out in terms of $R, r, d$ the results operating with $T_9^{-1}$ on some of the functions defined above. Once these have been calculated, we could almost instantaneously pass on from rational to irrational forms or vice versa. Remembering that

\[ 2a^2 = R - d + r, \quad 2a^2 = R + d + r, \quad 2a^2 = 2R, \]

it follows that:

\[ T_9^{-1} E_1 = -u^2 + v^2 + w^2 = R + d = \frac{1}{a} \]

\[ T_9^{-1} E_2 = u^2 - v^2 + w^2 = R - d = \frac{1}{b} \]

\[ T_9^{-1} E_3 = u^2 + v^2 - w^2 = r = \frac{1}{c} \]
We are now fully equipped to proceed to the calculation of a number of porisms.

\[ C_5(3) \] is given by

\[ t_3 = 1 \text{ or } a + b + c = 0 \text{ or } (R+d)^{-1} + (R-d)^{-1} = r^{-1} \]

On applying \( T_{e-1} \), we get Chaundy's irrational form

\[ \sqrt{(R-d)^{-1} + (R+d)^{-1}} = \sqrt{2R} \]

We have already seen that \( C_4(4) \) is given by \( R = d \) and \( C_1(4) \) by \( R = d = 0 \). \( K(4) \) is given by \( t_4 = 1 \) or \( a^2 + b^2 = c^2 \) or \( E_2 = 0 \). \( C_4(8) \) is given by \( T_e \) on \( R + d = 0 \) leading to \( E_2 = 0 \) and hence \( C_4(8) \equiv E_2 = 0 \).

Hence \( C_4(8) \) is given by \( t_4 = 1 \) which leads to \( f_4 = 0 \).

\[ \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 2abc = 0 \]

The various ways of writing \( f_4 = 0 \) are

\[ (a + b)(b + c)(c + a) = a^2 + b^2 + c^2, \]
\[ (a + b)(b + c) = d(a^2 + b^2 + c^2), \]
\[ (a - b + c)(a - b - c) + 6abc = 0, \]
\[ a(-a^2 + b^2 + c^2) + b(a^2 - b^2 + c^2) + c(a^2 + b^2 - c^2) + 2abc = 0, \]
\[ e_{12} + e_{13} + e_{23} + e_{4} = 0, \text{ etc.} \]

\( K(6) \) is given by \( T_e \) on \( C_1(3) \) or \( T_e \) on \( e_4 \) or \( E_2 = 0 \). \( T_e \) on \( C_3(3) \) gives \( e_6 \), \( e_7 \), \( e_8 \) which gives nothing new. \( K(6) \) may also be written

\[ E_{2;1} + E_{2;1}^{-1} = E_{2;1}^{-1} \]

\( C_1(13) \) is given by \( T_e \) on \( C_1(6) \) or \( F_2 = 0 \). \( C_1(13) \) is accordingly \( F_2 = 0 \). \( C_4(7) \) is given by \( t_4 = 1 \) giving \( g_4 = 0 \). Hence

\[ C_4(7) \equiv g_4 = 0, \quad C_4(14) \equiv g_4 = 0 \quad \text{and} \quad C_4(14) \equiv g_1 = 0. \]

\[ T_{e-1} t_4 = 0 \] gives

\[ a + b + c + \frac{1}{2} + 2abc = 0 \]
which is Chaundy's irrational representative and which excites admiration when compared with the corresponding representative for a pentagon. It follows that the porism for a heptagon can be exhibited in each of the six rational forms corresponding to the six forms given above for a pentagon.

\[ K(9) \text{ is given by } T_9 \text{ on } K(4) \text{ or } G_9 = 0 \text{ or } E_1^{-1} + E_2^{-1} = E_3^{-1}. \]

\[ C_9(10) \text{ is given by } T_9 \text{ on } C_9(8) \text{ or } G_9 = 0 \text{ or } E_1^{-1} + E_2^{-1} = E_3^{-1}. \]

\[ C_9(10) \text{ is given by } T_9 \text{ on } C_9(8) \text{ or } G_9 = 0 \text{ or } E_1^{-1} + E_2^{-1} = E_3^{-1}. \]

\[ C_9(0) \text{ may be derived from } T_9 \text{ on } C_9(0) \text{ or } h_9 = 0. \]

Hence \( C_9(0) \equiv h_9 = 0, \quad C_9(18) \equiv h_2 = 0, \quad \text{and} \quad C_9(18) \equiv h_1 = 0. \)

\( h_9 \) may also be exhibited in the form

\[ a + b \frac{F_1}{F_1} + c \frac{F_2}{F_2} = 0, \]

which is changed by \( T_9^{-1} \) into Chaundy's irrational representative, viz.,

\[ \sqrt{(R-d+R)} \left( \frac{R+d+r}{R^2-d^2-2rd} \right) + \sqrt{(R+d)} = 0. \]

\[ K(10) \text{ may be best obtained by } T_9 \text{ on } j_9 = 0 \text{ which turns into} \]

\[ H_9 = 0 \text{ or } \frac{G_9 + G_9}{E_1} + \frac{G_9 + G_9}{E_2} + 2E_1E_2E_9 = 0. \]

\[ \text{Accordingly,} \]

\[ C_9(20) \equiv h_9 = 0, \quad C_9(20) \equiv h_9 = 0. \]

\( T_9 \text{ on } j_9 = 0 \text{ gives } h_9 = 0 \text{ or } f_9 = f_9 = 0 \text{ which is nothing new.} \)

We come next to the heptagon, which is given by \( i_9 = 1 \text{ or } f_9 = f_9 = f_9 = f_9 = 0 \).

This may be rewritten in the form

\[ aE_9 + bE_9 + cE_9 = 2ab \frac{E_9}{F_1} \quad \text{or} \quad \frac{aE_9}{F_1} + \frac{bE_9}{F_2} + \frac{cE_9}{F_2} = 0. \]

\( \text{Hence } C_9(11) \equiv i_9 = 0 \text{ or } \)

\[ \frac{aE_9}{F_1} + \frac{bE_9}{F_2} + \frac{cE_9}{F_2} = 0. \]

On applying \( T_9^{-1} \) we obtain the irrational representative

\[ \frac{aE_9}{F_1} + \frac{bE_9}{F_2} + \frac{cE_9}{F_2} = 0. \]

\( \text{or more explicitly} \)

\[ \frac{(R+d+r)(R+d+r)}{R^2-d^2-2rd} \left( \frac{R+d+r}{R^2-d^2-2rd} \right) + \sqrt{(R+d)} = 0. \]

\( \text{This may be compared with Chaundy's result. It is obvious that Chaundy's result contains two errors. In the denominators of the first two terms of the portim as} \}

exhibited above, Chaundy's porism contains \(+2Rd\) and \(-2Rd\), whereas these ought to be \(+2Rd\) and \(-2Rd\) respectively. With this modification, Chaundy's result tallies completely with the one given here. That these represent actual errors in calculation and not merely mistakes follows from an inspection of \( C(22), K(22) \) and \( C(44), \) all of which are tainted by the same two errors in \( C(11). \) Consequently, Chaundy's \( K(44) \)--which he considers unsuitable--must contain the same errors, as also every other porism deduced by him from \( C(11). \)

\( \text{It follows that } C_9(22) \equiv j_9 = 0 \text{ and } C_9(22) \equiv j_9 = 0. \)

\( \text{We exhibit } C_9(22) \text{ in the irrational form so that comparison may be made with Chaundy's result:} \)

\[ \left( R+d+r \right)^{2} \left( R+d+r \right) \frac{(d-r)}{R^2-d^2-2rd} + \sqrt{(R+d)}} = 0. \]

\( \text{The correct forms of } C(11) \text{ and } C(22), \text{ both rational and irrational, appear, I believe, for the first time in this paper.} \)

\( \text{The majority of the porisms which follow are believed to have been given here for the first time in their explicit forms. We proceed further. } K(12) \text{ would result from } T_9 \text{ on } K(8) \text{ or} \]

\[ T_9F_9 = 0 \text{ or } j_9 = 0 \text{ or } G_9^{-1} + G_9^{-1} = G_9^{-1}. \]

\[ C_9(24) \equiv i_9 = 0 \text{ and } C_9(24) \equiv i_9 = 0. \]

The porism of a 18-gon is furnished by \( t_9j_9 = 1 \text{ leading to} \)

\[ (h-c)F_9F_9 = 0 \]

Hence \( \text{or } j_9 = 0, \quad C_9(30) \equiv j_9 = 0 \text{ and } C_9(30) \equiv j_9 = 0 \)

\( j_9 = 0 \text{ may also be written as} \)

\[ \frac{aG_9 + 4bG_9 + 4abF_9}{F_9} = 0. \]

On applying the irrationalizing transformation, this becomes

\[ \frac{uE_9 + vE_9 + wE_9}{F_9} + \frac{uE_9 + vE_9 + wE_9}{F_9} = 0. \]

or more explicitly

\[ \frac{\sqrt{(R^2-d^2+R^2)} (R-d+r)}{R^2-d^2+R^2} + \sqrt{(R+d)} = 0. \]

\( \text{We next attack the 14-gon. } K(14) \text{ will be given by either } t_9j_9 = 1 \text{ or by } T_9 \text{ on } C(7) \text{ or by } T_9j_9 = 0 \text{ or } j_9 = 0. \)

\( \text{Accordingly} \)

\[ C_9(28) \equiv j_9 = 0 \text{ and } C_9(28) \equiv j_9 = 0. \]

\( K(14) \text{ may be exhibited in the form} \)

\[ \frac{E_9 + E_9}{G_9} = \frac{1}{E_9} \quad \text{or} \quad E_9 + E_9 = 0. \]
The hurdle provided by a 15-agon is more troublesome. Chaundy just clears it, though not too gracefully, in the second attempt. The source of trouble springs from the two degenerate relatives, the triangle and the pentagon, who manage to creep in singly or together, no matter how we attack the problem. If we set $t_4 = 1$, the triangle and the pentagon both creep in along with the 15-agon; if we set $t_5 = 1$, the triangle alone is present and if we set $t_4 = 1$, the pentagon keeps company with the 15-agon. If we apply $T_5$ on $C_5$, the pentagon is present even if we know the transformation $T_5$. $T_5$ on $C_5$ would bring in the undesirable triangle. One might have been tempted to give up the 15-agon as hopeless and to run away from the heavy algebra which blocks the way. But Chaundy's result gives rise to a hope that the algebra would simplify tremendously, if gracefully performed, and so we gather courage and proceed.

We set $t_4 = 1$, which may be considered to result out of $T_5$ on $t_4 = 1$. The result simplifies to

$$aH_4H_5 = (b - c)H_5H_4,$$

which may be rewritten as

$$aH_4H_5 + bH_5H_4 = 0.$$

The symmetric portion must, therefore, run

$$aH_4H_5 + bH_5H_4 + cH_5H_4 = 0,$$

a symmetric expression of degree 23 in $a$, $b$, $c$; the redundant factor $a + b + c$ still remains to be worked out. Before we plunge into the algebra, we note that

$$a + b + c = 0,$$

$$H_4 + H_5 + H_4 = 0,$$

could also have resulted out of $T_5$ on $a + b + c = 0$, and accordingly $T_5$ must be given by

$$a \rightarrow H_4, b \rightarrow H_5, c \rightarrow H_4.$$

We have here stumbled across a gun, which we were not looking for. Besides providing us with a tool far more powerful than $T_5$, it gives us a clue to the general law of repulsion that $p$ is a prime exceeding 2. We have only to note that $T_5$ implies $a \rightarrow cK(0)$ and that $T_4$ involves $a \rightarrow cE(0)$ to conjecture that in general $T_p$ should involve $a \rightarrow c/K(0)$. We shall have occasion later on to test this conjecture for $p = 7$, when we come to $C(2)$. On trying to remove the factor $a + b + c$, we find

$$aH_4H_5 + bH_4H_5 = 0,$$

hence $C_5(15)$ is given by $k_4 = 0$. Also $C_5(15) = k_4 = 0$. $C_5(20)$ = $k_4 = 0$. $C_5(30)$ = $k_4 = 0$.

Now

$$k_4 = a \rightarrow c,H_4 = 0,$$

$$aH_4H_5 - bH_5H_4 - 2bH_5H_4 = 0,$$

$$C_5(15) = aH_4H_5 - bH_5H_4 - 2bH_5H_4 = 0.$$

The 17-agon yields the portion on setting $t_4 = 1$. This gives

$$C_4(17) = 4a(c - b)cH_4H_5,\quad B_5H_5, C_5G_5, G_5 = 0.$$

$$C_4(19) = 4a(c - b)cH_4H_5,\quad B_5H_5, C_5G_5, G_5 = 0.$$

$$C_4(33) = 4a(c - b)cH_4H_5,\quad B_5H_5, C_5G_5, G_5 = 0.$$

$$C_4(84) = 4a(c - b)cH_4H_5,\quad B_5H_5, C_5G_5, G_5 = 0.$$

Hence

$$k_4 = 0.$$

The next polygon is the 17-agon. This is given by $t_4 = 1$ leading to

$$C_4(19) = (b - c)cH_4H_5,\quad H_4H_5, C_5G_5, G_5 = 0.$$

$$C_4(84) = (b - c)cH_4H_5,\quad H_4H_5, C_5G_5, G_5 = 0.$$

$$T_5$ on $C_5(19)$ implies $T_5$ on $H_5 = 0$. Hence

$$k_4 = 0.$$
which simplifies to

\[ \alpha J_x + (\alpha - \beta)J_y J_z = 0 \]

or

\[ (\alpha + \beta - \gamma) = 0 \]

This could have been deduced by applying \( T_1 \) on \( \alpha + \beta - \gamma = 0 \) for a triangle. Hence \( T_1 \) is defined by

\[ a \rightarrow \alpha, \quad b \rightarrow \beta, \quad c \rightarrow \gamma \]

which corroborates our conjecture regarding the general law of transformation for replicating a prime number of times.

As in the case of a 15-gon, the porism is tainted by the presence of an irrelevant factor \( a + b + c \) corresponding to a triangle. Now

\[ aJ_x + bJ_y + cJ_z = (a + b + c) [ - 4E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z ] \]

Hence \( C_{(22)} \) is given by,

\[ 4E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z = 0 \]

\( C_{(23)} \) is defined by

\[ 4E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z = 0 \]

\( C_{(24)} \) is defined by

\[ 4E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z = 0 \]

The next porism concerns a 22-gon. \( T_2 \) on \( l_2 = 0 \) gives \( K_{(22)} \) in the form

\[ (G_x + G_y - \frac{2E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z}{E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z} = 0 \]

Hence

\[ C_{(44)} = \frac{G_x + G_y - \frac{2E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z}{E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z} = 0 }{E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z} \]

and

\[ C_{(44)} = - \frac{G_x + G_y - \frac{2E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z}{E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z} = 0 }{E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z} \]

On setting \( l_1 l_2 = 1 \), we obtain \( C_{(23)} \) in the form

\[ 2a(c - b)E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z = 0 \]

from which

\[ C_{(23)} = 2a(c - b)E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z = 0 \]

\( C_{(44)} = 2a(c - b)E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z = 0 \]

and

\[ C_{(44)} = - 2a(c - b)E, F, K, G, L, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z, J_x, J_y, J_z = 0 \]

\( K_{(24)} \) is defined by

\[ T_1 T_2 = 0 \] giving

\[ K_{(24)} = 1 + K_{(23)} = 0 \]

This is homogeneous and symmetric in \( a, b, c \) and of degree 13815. Its irrational representative can be worked out without much trouble, if desired. \( C_{(121)}, C_{(243)} \) and \( C_{(432)} \) follow readily on proper sign adjustments.

If our ambition had been a little more ambitious, we would have aimed at \( C_{(161)} \). The method would have been to calculate \( l_1 \) by \( T_2 \) on \( l_3 \) and \( l_2 \) by \( T_2 \) on \( l_3 \). Thus \( l_1 l_2 \) equated to 1 would yield \( C_{(44)} \), from which \( C_{(44)} \) would immediately be written down. Then \( T_2 \) on \( C_{(44)} \) would yield \( K_{(22)} \), from which \( C_{(121)}, C_{(243)} \) would come out by proper sign adjustments. Then \( T_1 \) is defined by

\[ a \rightarrow a', \quad b \rightarrow b', \quad c \rightarrow c, \quad \alpha \rightarrow K_{(24)} \]
Considerations of completeness impel us to introduce elliptic functions at this conclusion stage. The treatment is in principle suggested by Clifford's work on two-to-one correspondences. Let

\[ f(t, t') = a(b + c)(1 + t + t'^2) - 4bc(t^2 + t'^2) + b + c(t^2 + t'^2) = 0 \]

\[ \therefore \quad dt \int \frac{\partial f}{\partial t'} + dt' \int \frac{\partial f}{\partial t} = 0 \]

or

\[ \frac{dt}{\sqrt{1 + 2a(t^2 - 1)}} - \sqrt{1 + 2a(t'^2 - 1)} \]

where

\[ \lambda = 1 + \frac{2a}{(t^2 - 1)} \]

We assume \( \lambda > 1 \) and set

\[ \lambda = \text{real } \omega, \quad \frac{dx}{\sqrt{1 + (t^2 - 1)^2}} \quad \text{and} \quad \frac{dx'}{\sqrt{1 + (t'^2 - 1)^2}} \]

This leads to

\[ \int \frac{dx}{\sqrt{(1 - \omega^2)(1 - \omega'\omega)}} = \int \frac{dx'}{\sqrt{(1 - \omega^2)(1 - \omega'\omega)}} \]

where \( \omega = 1 - \frac{t^2}{\omega^2} \), yielding

\[ \omega^{-1}(x, k) - \omega^{-1}(x', k) = \text{constant} \]

\[ \text{Suppose } \omega' = 0, \text{gives } x = x_n. \text{Hence} \]

\[ \omega^{-1}(x, k) - \omega^{-1}(x, k) = \omega^{-1}(x_n, k) \]

Since \( x_n = 0 \),

\[ \omega^{-1}(x, k) = \omega^{-1}(x, k) = n = \int_0^1 \frac{dt}{\sqrt{1 - t^2}} \sin \theta \]

as the condition of porism.

Hence

\[ x_n = \omega^{-1}(x_n, k) = \sqrt{\frac{1 + \omega^2}{1 + \omega^2}} \]

\[ \therefore \quad \omega \left\{ \int x n \omega^{-1}(x_n, k) \right\} \sqrt{\lambda + 1} \sqrt{\lambda + 1} \sqrt{\lambda - 1} \]

or

\[ a \left\{ \frac{1}{n} \right\} \sqrt{b^2 - a^2} + b \sqrt{c^2 - a^2} \]

\[ \frac{a}{c + b} \]

Obviously, this cannot be the simplest condition for the porism of an \( n \)-agon in general, unless \( n \) be prime.
All porisms are homogeneous in $a, b, c$. It is noticed that $C(8)$ is of degree 1, $C(6)$ of degree 3, $C(7)$ of degree 0 and in general $C(p)$ is of degree $\frac{1}{2}(p^2 - 1)$ for prime $p$. If $n$ is a product of two distinct primes $p$ and $q$, each different from 2, then the porism corresponding to $p$ and $q$ factorizes as irrelevant factors. Hence the class porism $C(pq)$ will be homogeneous in $a, b, c$ and of degree $\frac{1}{2}[(p^2q^2 - 1) - (p^2 - 1)(q^2 - 1)]$ or $\frac{1}{2}[(p^2 - 1)(q^2 - 1)]$. However, if $n$ is a prime square, the degree of $C(p^2)$ is $\frac{1}{2}[(p^2q^2 - 1) - (p^2 - 1)] = \frac{1}{2}p(p^2 - 1)$. Now since $C(p)$ is of degree $\frac{1}{2}(p^2 - 1)$, $K(p^2)$ will be of degree $\frac{1}{2}(p^2 - 1)$, and hence the transformation $T_p$ will multiply the degree of $C(p)$ by $\frac{1}{2}(p^2 - 1)$. Hence $T_p$ on $C(p^2q^2 - 1)$ by $p^2q^2 - 1$. Hence $T_p$ on $C(p^2)$ is of degree $\frac{1}{2}(p^2q^2 - 1)$ or $\frac{1}{2}p^2(p^2q^2 - 1)$. It follows that $C(q^2)$, given by $T_p$ on $C(q^2)$, contains no irrelevant factors.

By similar considerations, it follows that $C(pqr)$, where $p, q, r > 2$ and prime, would be of degree

$$\frac{1}{2}[(p^2q^2r^2 - 1) - (p^2 - 1)(q^2 - 1)(r^2 - 1)] = \frac{1}{2}p^2q^2r^2(p^2 - 1)(q^2 - 1)(r^2 - 1).$$

Now $T_p$ on $C(q)$ is of degree $\frac{1}{2}p^2(q^2 - 1)$ and hence contains a factor of degree $\frac{1}{2}(q^2 - 1)$ which corresponds to $C(q)$. Hence

$$T_p$ on $C(pq) = C(q) C(pq).$$

Similarly

$$T_p$ on $C(p) = C(p) C(pq).$$

We came across this phenomenon when discussing $C(15)$ and $C(21)$. But $T_p$ on $C(pq)$ is of degree $\frac{1}{2}p^2(q^2 - 1)$ and hence contains no irrelevant factors. In general, it is found that

$$T_p^{-1} T_q^{-1} C(pqr) = C(p^2 q^2 r).$$

and contains no irrelevant factors.

As the transformation $T_p$ multiplies the dimension by 4, the counting of dimensions for even porisms presents no problem whatsoever. The table given on page 108 gives the dimensions of porisms for an $n$-agon for $3 \leq n \leq 254$. An interesting fact, capable of a simple proof, emerges from the table. If the dimension of a porism is odd, the polygon has $p^2$ sides, where $p$ is prime and $r$ a positive integer.

The full invariant forms for the porisms in terms of $\Delta$, $\Delta'$, $\Theta$, $\Theta'$ may be obtained by the transformation

$$\Delta = \frac{\Theta}{2hr^2 + r^2 - d^2} = \frac{\Theta'}{h^2 + 2r^2 - d^2} = \frac{\Delta'}{r^2}$$

but like Chaudhary, I am one of those who prefer to see the porisms exhibited in circular terms.

References

2. Cayley, Phil. Mag., VI, 1859, pp. 29-32.
10. Darboux, G., Principes de géométrie analytique, livre III, 1897, p. 287.