$$\frac{n!}{e^n}\left[1-\frac{1}{n}-\frac{1}{2n^2}+\cdots\right],$$

but the series has been extended by me (1947) as follows:

$$\frac{n!}{e^n}\left[1-\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\frac{49}{24n^4}+\frac{629}{120n^5}+\frac{6961}{720n^6}+\frac{10189}{2520n^7}\right.$$

$$\left.-\frac{693229}{13440n^8}-\frac{13805253}{302880n^9}-\frac{648211309}{9628800n^{10}}-\cdots+\cdots\right].$$

It is instructive to compare this last result with (7). The two asymptotic expansions are identical as far as the fourth term, but differ from the fifth term onwards. This throws interesting light on some conjectures of Erdös and Kaplansky about the asymptotic number of the general n by k Latin rectangle. I hope to examine the implications in a subsequent paper.

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PONCELET PORISM IN TWO CIRCLES

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INTRODUCTION

The general problem of investigating the condition that an n-agon inscribed in a given conic S should circumscribe any other given conic s has in some form or the other received the attention of a galaxy of prominent mathematicians including Euler, Steiner [16] and Cayley [1], [2], [3]. The general porisms in terms of the invariants and covariants of S and s rapidly become unmanageable and do not appear to have claimed any great sustained interest. It is otherwise when S and s are considered to be respectively reduced to homographically equivalent circles of radii R and r with centres a distance d apart. Weill [18], Titchmarsh [17], Rosanes and Pasch [15], Chaundy [5], [6] and numerous others have delved by such methods into the secrets of a circle-pair, but the palm appears to me to have been carried away by Chaundy, who in his two papers has veritably unearthed a priceless treasure.

In his first paper, Chaundy shows that the porism of a $2n$-agon, when purged of all irrelevant factors, splits into two factors, one simplicity itself in comparision with the other. In conformity with Chaundy's nomenclature and subsequent notation, the bulkier factor will be said to correspond to the porism of the "inseparable" case and will be denoted by $K(2n)$. Of the more docile factor, corresponding to the "separable" case, there always happen to be two forms. If $2n = 2^r(2m+1)$, these two conditions can be associated with the two forms for the porism of a $(2m+1)$-agon, and in conformity with what follows, these will respectively be denoted by $C_1(2n)$ and $C_2(2n)$, or simply by $C(2n)$ when either may be implied. When all redundant factors have been removed, two forms occur of a porism of a $(2n+1)$-agon, one corresponding to the case when s is the ex-circle and the other to the case when s is the in-circle. These will be denoted respectively by $C_1(2n+1)$ and $C_2(2n+1)$, the suffix being dropped when either may be implied. Chaundy proves that to derive $C(4n+2)$ from $C(2n+1)$, an interchange of R and d suffices. Using elliptic functions throughout and converting odd porisms into even by interchanging R and d, he exhibits in their smartest attire the following porisms: $C(3)$, $C(4)$, $C(5)$, $C(6)$, $C(7)$, $C(8)$, $C(9)$, $C(10)$, $C(12)$, $C(14)$, $C(16)$, $C(18)$ and $C(20)$. $C(15)$ and $C(30)$ which also occur are in company of one inferior porism each. This is a weakness to which the great Steiner also succumbed when he held up his $C(5)$ for display, and the error has been pointed out by Chaundy. In fairness, however, to Chaundy, it must be pointed out that the errors in $C(15)$ and $C(30)$ have been rectified in the second paper, where he gives in much less explicit forms $C(15)$, $C(21)$, $C(33)$ and $C(35)$. The irrational representatives of these have not been given in terms of R, r and d—what was Chaundy's objective in the first paper.

Towards the end of his work, Chaundy obtains $C(11)$, $C(22)$, $K(22)$ and $C(44)$. All these expressions are erroneous, as will be shown subsequently. Chaundy concludes his work with the remark: "I have calculated $K(44)$, but it is unprintable." However, as $K(44)$ could only have been calculated from $C(11)$ as a starting-point, the same remark applies to it as to $C(11)$. Chaundy also claims to have scored a century by evaluating $C(112)$. But he complains that the task was a heart-rending one and the final result unprintable.

The purpose of the present investigation is to establish as many porisms as possible in a systematic manner without any use of elliptic functions and by the comparatively elementary methods of the theory of a two-two correspondence. The total number of porisms that can be simply and elegantly derived by these methods transcends the wildest hopes with which I set out upon the task. Many of the results given here, I believe, go far beyond the total ground covered by all the previous writers on the subject. A variety of methods have been developed by the previous writers, but these suffered from the defect that while some yielded the porisms in irrational forms, others yielded them in their rational forms, and the task of transforming the one form into the other was in general a heart-breaking one. The result has been that mathematicians have regarded some one form with greater favour than the other. Chaundy, for example, is all for the irrational form, and he does not appear to be too pleased with the rational porism of an enneagon obtained by Richelot [14]. This defect may now be considered to be removed by the present paper, as the passage from one form to the other may be effected with the same ease with which one may switch on or off an electric light. Another feature, which I believe deserves mention, is the discovery by tentative methods of the general replicating law in its explicit form. Chaundy sets up explicit machinery for duplication alone, but the general approach lies concealed in the formulae for inseparable cases alone, and as Chaundy hardly displays any great interest in these particular cases, he appears to have missed this treasure.

It will be found, however, that elliptic functions do occur towards the end of the paper. But it must be mentioned that they have nothing to do with the main purpose of the paper, and the idea in bringing them in is only to give a completeness to the whole theory, which it would otherwise lack. I must confess, however, that I am deeply indebted for many of the ideas to the works of Weill [18], Clifford [6], [7], [8], Darboux [9], [10] and Titchmarsh [17] on the theory of a two-two correspondence, with which the problem is inextricably bound. But my sincerest thanks reach out to Chaundy whose two papers stood out as guiding stars, whenever I was about to despair.

1

As already mentioned, R will denote the radius of the circum circle S, r the radius of the in- or ex-circle s and d the distance separating their centres. Let

$$2u^2 = R-d+r, \qquad 2v^2 = R+d+r, \qquad w^2 = R$$

and

$$a^{-1} = R+d, \qquad b^{-1} = R-d, \qquad c^{-1} = r.$$

If the chord joining the points $A_i(R\cos\theta_i,\ R\sin\theta_i)$ and $A_{i+1}(R\cos\theta_{i+1},\ R\sin\theta_{i+1})$ on the circle $S \equiv x^2+y^2-R^2 = 0$ touches the circle $s \equiv (x-d)^2+y^2-r^2 = 0$, we must have

$$R\cos\tfrac{1}{2}(\theta_{i+1}-\theta_i) - d\cos\tfrac{1}{2}(\theta_{i+1}+\theta_i) = \pm r,$$

the sign to be taken positive if s be the in-circle, negative otherwise. In what follows, we shall choose the positive sign throughout and write $-r$ for r or $-c$ for c to derive $C_1(n)$ from $C_2(n)$ or vice versa. On writing $t_i = \tan\tfrac{1}{2}\theta_i$, the condition of contact may be rewritten as

$$u^2(t_i+t_{i+1})^2 + v^2(1-t_it_{i+1})^2 = w^2(1+t_it_{i+1})^2. \tag{1.1}$$

This is the quadratic involution connecting the points A_i, A_{i+1}. Assuming the poristic nature of the problem, no loss of generality occurs by setting $\theta_0 = 0$, giving $t_0 = 0$ and $t_1^2u^2 = w^2-v^2$. Choosing the positive sign for t_1, we may derive from it two values for t_2, one of which will be t_0. Rejecting the latter value, we get a unique value for t_2. If now the values of $t_0, t_1, t_2, \ldots t_k$ be unique the quadratic involution will determine two values for t_{k+1}, one of which will be t_{k-1}. Rejecting the latter value, t_{k+1} is uniquely determined. This establishes the induction, leading to unique values for each of $t_0, t_1, t_2, \ldots t_i$ for all integral values of i.

The condition that $A_{i-1}A_i$ should touch s is represented by

$$u^2(t_i+t_{i-1})^2 + v^2(1-t_it_{i-1})^2 = w^2(1+t_it_{i-1})^2 \tag{1.2}$$

From (1.1) and (1.2) it follows that

$$t_{i+1}+t_{i-1} = \frac{-u^2+v^2+w^2}{u^2}\cdot\frac{2t_i}{1-t_i^2t_1^2} \tag{1.31}$$

and

$$t_{i+1}t_{i-1} = \frac{t_i^2-t_1^2}{1-t_i^2t_1^2} \tag{1.32}$$

so that

$$\frac{t_{i+1}+t_{i-1}}{1-t_{i+1}t_{i-1}} = \frac{-u^2+v^2+w^2}{u^2-v^2+w^2}\cdot\frac{2t_i}{1-t_i^2}. \tag{1.33}$$

If we now eliminate t_i between (1.1) and (1.2), we obtain

$$a^2(t_{i+1}+t_{i-1})^2 + b^2(1-t_{i-1}t_{i+1})^2 = c^2(1+t_{i-1}t_{i+1})^2. \tag{1.4}$$

From the similarity of (1.1) and (1.4), it follows that replacing u, v, w by a, b, c in the three equations (1.3) would lead to

$$t_{i+2}+t_{i-2} = \frac{-a^2+b^2+c^2}{a^2}\cdot\frac{2t_i}{1-t_i^2t_2^2} \tag{1.51}$$

$$t_{i+2}t_{i-2} = \frac{t_i^2-t_2^2}{1-t_i^2t_2^2} \tag{1.52}$$

and

$$\frac{t_{i+2}+t_{i-2}}{1-t_{i+2}t_{i-2}} = \frac{2t_i}{1-t_i^2}\cdot\frac{-a^2+b^2+c^2}{a^2-b^2+c^2}. \tag{1.53}$$

From a comparison of (1.1) and (1.4), it follows further that to convert t_i into t_{2i} any one of the following equivalent transformations would suffice:

$$u^2 \to a^2, \qquad v^2 \to b^2, \qquad w^2 \to c^2;$$
$$u^2 \to (-u^2+v^2+w^2)^{-2}, \qquad v^2 \to (u^2-v^2+w^2)^{-2}, \qquad w^2 \to (u^2+v^2-w^2)^{-2};$$
$$a \to (-a^2+b^2+c^2)^{-1}, \qquad b \to (a^2-b^2+c^2)^{-1}, \qquad c \to (a^2+b^2-c^2)^{-1};$$
$$R \to c^2, \qquad r \to (a^2+b^2-c^2), \qquad d \to (b^2-a^2).$$

This is the general duplicating transformation which we denote by T_2. If T_2 operates on any homogeneous function of a, b, c of degree i, it is converted into another homogeneous function of a, b, c of degree $4i$. The form $u^2 \to a^2$, $v^2 \to b^2$, $w^2 \to c^2$, however, requires care in operation. If the porism on which it operates is a rational function of u^2, v^2, w^2, duplication is effected without fail. But if the porism is in its simplest irrational form, which is the form preferred by Chaundy, duplication does not always take place on applying the transformation. Four cases may be distinguished:

$$T_0 \equiv u \to a, \qquad v \to b, \qquad w \to c$$
$$T_0' \equiv u \to -a, \qquad v \to b, \qquad w \to c$$
$$T_0'' \equiv u \to a, \qquad v \to -b, \qquad w \to c$$
$$T_0''' \equiv u \to a, \qquad v \to b, \qquad w \to -c.$$

In practice, it is readily found that T_0 only rationalizes the porism and does nothing else; T_0' rationalizes, interchanges R and d and converts r into $-r$; T_0'' rationalizes and interchanges R and d; and finally that T_0''' rationalizes and changes r into $-r$. These transformations will prove their worth in the subsequent discussion.

Repeating on (1.4) the operation by which (9.4) was derived from (1.1) and by merely using T_2, we obtain

$$(-a^2+b^2+c^2)^{-2}(t_{i+2}+t_{i-2})^2+(a^2-b^2+c^2)^{-2}(1-t_{i+2}t_{i-2})^2 = (a^2+b^2-c^2)^{-2}(1+t_{i+2}t_{i-2})^2 \quad (1.6)$$

from which follows

$$t_{i+1}+t_{i-1} = \frac{2t_i}{1-t_i^2t_i^2}\cdot\frac{\{(-a^2+b^2+c^2)^4-16b^2c^2(b^2-a^2)(c^2-a^2)\}}{(a^2-b^2+c^2)^2(a^2+b^2-c^2)^2} \quad (1.71)$$

$$t_{i+1}.t_{i-1} = \frac{t_i^2-t_i^2}{1-t_i^2t_i^2} \quad (1.72)$$

and

$$\frac{t_{i+1}+t_{i-1}}{1-t_{i+1}t_{i-1}} = \frac{(-a^2+b^2+c^2)^2-16b^2c^2(b^2-a^2)(c^2-a^2)}{(a^2-b^2+c^2)^2-16c^2a^2(a^2-b^2)(c^2-b^2)}\cdot\frac{2t_i}{1-t_i^2} \quad (1.73)$$

It follows that the transformation which converts t_r into t_{ir} is $T_2\,T_2$ or T_2^2 or T_4 equivalent to

$$a \to (-3a^4+b^4+c^4-2b^2c^2+2c^2a^2+2a^2b^2)^{-1}, \text{ etc.}$$

We next seek to eliminate t_{i+1} between

$$u^2(t_i+t_{i+1})^2+v^2(1-t_it_{i+1})^2 = w^2(1+t_it_{i+1})^2$$

and

$$a^2(t_{i+1}+t_{i+3})^2+b^2(1-t_{i+1}t_{i+3})^2 = c^2(1+t_{i+1}t_{i+3})^2,$$

reaching (after wading through a mass of algebra) the objective in the form

$$\{u^2(t_i+t_{i+3})^2+v^2(1-t_it_{i+3})^2-w^2(1+t_it_{i+3})^2\}\left\{\frac{a(b+c)(t_i+t_{i+3})^2}{(-a+b+c)^2}+\frac{b(c+a)(1-t_it_{i+3})^2}{(a-b+c)^2}-\frac{c(a+b)(1+t_it_{i+3})^2}{(a+b-c)^2}\right\} = 0.$$

As the first factor would imply a retracing in the path $A_0\,A_1\,A_2\ldots$, we reject it. The required eliminant may be written as

$$\frac{a(b+c)(t_i+t_{i+3})^2}{(-a+b+c)^2}+\frac{b(c+a)(1-t_it_{i+3})^2}{(a-b+c)^2} = \frac{c(a+b)(1+t_it_{i+3})^2}{(a+b-c)^2}. \quad (1.8)$$

It follows that the triplicating transformation T_3 converting t_r into t_{3r} is given by any one of the following equivalent forms:

$$u^2 \to \frac{u^2}{(-a+b+c)^2}, \text{ etc.,} \qquad u^2 \to \frac{a(b+c)}{(-a+b+c)^2}, \text{ etc.,}$$

$$a \to a(-3a^4+b^4+c^4-2b^2c^2+2c^2a^2+2a^2b^2)^{-1}, \text{ etc.}$$

From the involution (1.8), it follows that

$$t_{i-3}+t_{i+3} = \frac{\dfrac{-a(b+c)}{(-a+b+c)^2}+\dfrac{b(c+a)}{(a-b+c)^2}+\dfrac{c(a+b)}{(a+b-c)^2}}{\dfrac{a(b+c)}{(-a+b+c)^2}}\cdot\frac{2t_i}{1-t_i^2t_3^2} \quad \ldots (1.91)$$

$$t_{i-3}\,.\,t_{i+3} = \frac{t_i^2-t_3^2}{1-t_i^2t_3^2}, \quad \ldots (1.92)$$

and

$$\frac{t_{i-3}+t_{i+3}}{1-t_{i-3}t_{i+3}} = \frac{2t_i}{1-t_i^2}\cdot\frac{\dfrac{-a(b+c)}{(-a+b+c)^2}+\dfrac{b(c+a)}{(a-b+c)^2}+\dfrac{c(a+b)}{(a+b-c)^2}}{\dfrac{a(b+c)}{(-a+b+c)^2}-\dfrac{b(c+a)}{(a-b+c)^2}+\dfrac{c(a+b)}{(a+b-c)^2}} \quad \ldots (1.93)$$

The transformation T_6 is given by $T_3\,T_2$ or $T_2\,T_3$ and implies

$$u^2 \to \frac{a^2}{(-3a^4+b^4+c^4-2b^2c^2+2c^2a^2+2a^2b^2)^2}, \text{ etc.,}$$

The transformations T_5, T_7, T_{11}, etc. will be obtained by a different method in the subsequent work.

2

We have already established quadratic involutions connecting t_0, t_{i+1}; t_0, t_{i+2}; t_0, t_{i+3}; t_0, t_{i+4}. Let us suppose that further quadratic involutions have been established connecting t_0, t_{i+5}; t_0, t_{i+6}; ...; t_0, t_{i+r-1}. Now r may either be of the form $2s$ or of the form $2s+1$. If $r=2s$, the involution connecting t_0, t_{i+r} could be directly deduced

by operating with T_2 on the involution connecting t_i, t_{i+s}. If $r=2s+1$, we may eliminate t_{i+s} between the two involutions connecting respectively t_i, t_{i+s} and t_{i+s}, t_{i+r} and reject the retrogressing factor:

$$u^2(t_i+t_{i+r})^2+v^2(1-t_i t_{i+r})^2-w^2(1+t_i t_{i+r})^2.$$

Hence is obtained the involution connecting t_i, t_{i+r}. The induction is thus completed and we have proved the existence (for all values of r) of a quadratic involution of the form

$$a_r^2(t_i+t_{i+r})^2+b_r^2(1-t_i t_{i+r})^2=c_r^2(1+t_i t_{i+r})^2 \qquad \ldots (2.1)$$

It follows that

$$t_{i+r}\,.\,t_{i-r}=\frac{t_i^2-t_r^2}{1-t_i^2 t_r^2} \qquad \ldots (2.2)$$

for all values of i and r, provided we agree that $t_{-r}=-t_r$ and that $u^2\to a_r^2$, $v^2\to b_r^2$, $w^2\to c_r^2$ defines the general replicating transformation T_r.

3

From the initial assumption $\theta_0=0$, it follows that if an n-agon be inscribed in S and circumscribed about s, the n-agon will lie symmetrically about the line joining the centres of the two circles.

If $n=2m$, then θ_m would be an odd multiple of π and hence $t_m=\pm 1$. Since the polygon in this case is a genuine one, the medial diagonals must concur at a finite point and hence the porism given by $t_m^2=1$ can only be the porism corresponding to the "inseparable" case. Hence $K(2m)$ is simply given by $t_m^2=1$.

To obtain the porism of $(2m+1)$-agon, we need only remember that θ_m, θ_{m+1} must respectively be of the forms $(2r+1)\pi-\alpha$ and $(2r+1)\pi+\alpha$. It follows that the porism $C_2(2m+1)$ is given by $t_m\,t_{m+1}=1$. But from (2.2),

$$\frac{t_{m+1}^2-t_m^2}{1-t_{m+1}^2\,t_m^2}=t_1 t_{2m+1},$$

where the denominator vanishes but the numerator does not as $t_{m+1}^2=t_m^2$ would imply α to be a multiple of 2π, which is out of question. As t_1 is finite, it follows that $t_{2m+1}=\infty$ must also lead to the same porism.

We could also deduce the porism from the general condition $t_{m-j}\,t_{m+j+1}=1$. In each individual case, it will have to be decided which of these various conditions would lead to the porism most rapidly. But when $(2m+1)$ is prime, the simplest method is to set $t_m t_{m+1}=1$. When $(2m+1)$ is composite, the problem must become troublesome, as irrelevant factors are bound to enter into the porism and have to be removed at the cost of great labour, before the relevant porism is ready for exhibition.

Another point in this connection may also be mentioned. The porism thus obtained relates to the in-circle. The porism $C_1(2m+1)$ corresponding to the ex-circle is rapidly deduced by changing c into $-c$ in $C_2(2m+1)$.

We next come to the porism of a $(4m+2)$-agon for the "separable" case. The relevant condition can be deduced by considering only one half of the polygon, which lies symmetrically with respect to the common diameter. The situation in view can only arise if $\theta_m=\theta_{m+1}$ and consequently the polygon has to be retraced to the starting-point, and then to be continued on the other side of the bounding line. This implies that $t_m=t_{m+1}$ is the condition for the porism of a "separable" case. This gives $C_2(4m+2)$. The condition $C_1(4m+2)$ may be deduced from $C_2(4m+2)$ by merely changing the sign of c. The transformation T_2 can rapidly build up all porisms of the type $C(2^\gamma(2m+1))$. The only case that would not be covered by the argument would be the case $C(2^\gamma)$, but T_2 supplies all the relevant porisms if only $C(4)$ be known. To obtain $C(4)$, it is only necessary to observe that A_1 is one of the common points of s and S, and this can only happen if $R=\pm d$. The condition $R-d=0$ will be known as $C_2(4)$ and the condition $R+d=0$ as $C_1(4)$, although in this particular case the change of sign of c to convert the one form into the other becomes meaningless.

4

The various formulæ connecting the t's have been used to obtain the values of t_i up to and including $i=16$. To condense the formulæ, the following abbreviations have been used:

$$\begin{aligned}
e_0 &\equiv a+b+c,\\
e_1 &\equiv -a+b+c,\\
e_2 &\equiv a-b+c,\\
e_3 &\equiv a+b-c;\\
E_1 &\equiv -a^2+b^2+c^2,\\
E_2 &\equiv a^2-b^2+c^2,\\
E_3 &\equiv a^2+b^2-c^2.\\
f_0 &\equiv aE_1+bE_2+cE_3+2abc\\
f_1 &\equiv -aE_1+bE_2+cE_3-2abc\\
f_2 &\equiv aE_1-bE_2+cE_3-2abc\\
f_3 &\equiv aE_1+bE_2-cE_3-2abc\\
F_1 &\equiv -3a^4+b^4+c^4-2b^2c^2+2c^2a^2+2a^2b^2 \equiv -E_2E_3+E_3E_1+E_1E_2\\
F_2 &\equiv a^4-3b^4+c^4+2b^2c^2-2c^2a^2+2a^2b^2 \equiv E_2E_3-E_3E_1+E_1E_2\\
F_3 &\equiv a^4+b^4-3c^4+2b^2c^2+2c^2a^2-2a^2b^2 \equiv E_2E_3+E_3E_1-E_1E_2\\
F_0 &\equiv -a^4-b^4-c^4+2b^2c^2+2c^2a^2+2a^2b^2 \equiv E_2E_3+E_3E_1+E_1E_2 \equiv e_0e_1e_2e_3\\
g_0 &\equiv E_1E_2E_3+2abE_1E_2+2bcE_2E_3+2caE_3E_1\\
g_1 &\equiv E_1E_2E_3-2abE_1E_2+2bcE_2E_3-2caE_3E_1\\
g_2 &\equiv E_1E_2E_3-2abE_1E_2-2bcE_2E_3+2caE_3E_1\\
g_3 &\equiv E_1E_2E_3+2abE_1E_2-2bcE_2E_3-2caE_3E_1
\end{aligned}$$

$$G_1 \equiv -E_2^2E_3^2+E_3^2E_1^2+E_1^2E_2^2 \equiv (-a^2+b^2+c^2)^4-16b^2c^2(b^2-a^2)(c^2-a^2)$$

$$G_2 \equiv E_2^2E_3^2-E_3^2E_1^2+E_1^2E_2^2 \equiv (a^2-b^2+c^2)^4-16c^2a^2(c^2-b^2)(a^2-b^2)$$

$$G_3 \equiv E_2^2E_3^2+E_3^2E_1^2-E_1^2E_2^2 \equiv (a^2+b^2-c^2)^4-16a^2b^2(a^2-c^2)(b^2-c^2)$$

$$h_0 \equiv aF_2F_3+bF_3F_1+cF_1F_2$$

$$h_1 \equiv -aF_2F_3+bF_3F_1+cF_1F_2$$

$$h_2 \equiv aF_2F_3-bF_3F_1+cF_1F_2$$

$$h_3 \equiv aF_2F_3+bF_3F_1-cF_1F_2$$

$$H_1 \equiv -G_1E_2E_3+G_2E_3E_1+G_3E_1E_2-2E_1^2E_2^2E_3^2 \equiv \tfrac{1}{2}[F_1F_2F_3-F_0(-F_2F_3+F_3F_1+F_1F_2)]$$

$$H_2 \equiv G_1E_2E_3-G_2E_3E_1+G_3E_1E_2-2E_1^2E_2^2E_3^2 \equiv \tfrac{1}{2}[F_1F_2F_3-F_0(F_2F_3-F_3F_1+F_1F_2)]$$

$$H_3 \equiv G_1E_2E_3+G_2E_3E_1-G_3E_1E_2-2E_1^2E_2^2E_3^2 \equiv \tfrac{1}{2}[F_1F_2F_3-F_0(F_2F_3+F_3F_1-F_1F_2)]$$

$$H_4 \equiv G_1E_2E_3+G_2E_3E_1+G_3E_1E_2+2E_1^2E_2^2E_3^2 \equiv -f_0f_1f_2f_3$$

$$i_0 \equiv F_0(aE_1F_2F_3+bE_2F_3F_1+cE_3F_1F_2)+2abcF_1F_2F_3$$

$$i_1 \equiv F_0(-aE_1F_2F_3+bE_2F_3F_1+cE_3F_1F_2)-2abcF_1F_2F_3$$

$$i_2 \equiv F_0(aE_1F_2F_3-bE_2F_3F_1+cE_3F_1F_2)-2abcF_1F_2F_3$$

$$i_3 \equiv F_0(aE_1F_2F_3+bE_2F_3F_1-cE_3F_1F_2)-2abcF_1F_2F_3$$

$$I_1 \equiv T_2F_1 \equiv -G_2G_3+G_3G_1+G_1G_2$$

$$I_2 \equiv T_3F_2 \equiv G_3G_1-G_2G_1+G_1G_2$$

$$I_3 \equiv T_1F_3 \equiv G_2G_3+G_3G_1-G_1G_2$$

$$I_4 \equiv T_0F_0 \equiv F_0F_1F_2F_3$$

$$j_0 \equiv F_0(aG_1F_2F_3+bG_2F_3F_1+cG_3F_1F_2)+4abcE_1E_2E_3F_1F_2F_3$$

$$j_1 \equiv F_0(-aG_1F_2F_3+bG_2F_3F_1+cG_3F_1F_2)-4abcE_1E_2E_3F_1F_2F_3$$

$$j_2 \equiv F_0(aG_1F_2F_3-bG_2F_3F_1+cG_3F_1F_2)-4abcE_1E_2E_3F_1F_2F_3$$

$$j_3 \equiv F_0(aG_1F_2F_3+bG_2F_3F_1-cG_3F_1F_2)-4abcE_1E_2E_3F_1F_2F_3$$

$$J_1 \equiv -2E_1E_2E_3(-E_1G_2G_3+E_2G_3G_1+E_3G_1G_2)+G_1G_2G_3$$

$$J_2 \equiv -2E_1E_2E_3(E_1G_2G_3-E_2G_3G_1+E_3G_1G_2)+G_1G_2G_3$$

$$J_3 \equiv -2E_1E_2E_3(E_1G_2G_3+E_2G_3G_1-E_3G_1G_2)+G_1G_2G_3$$

$$J_4 \equiv 2E_1E_2E_3(E_1G_2G_3+E_2G_3G_1+E_3G_1G_2)+G_1G_2G_3 \equiv -g_0g_1g_2g_3$$

$$k_0 \equiv F_1F_2F_3c_1c_2c_3f_1f_2f_3-2E_1E_2E_3g_1g_2g_3$$

$$k_1 \equiv F_1F_2F_3c_0c_2c_3f_0f_2f_3-2E_1E_2E_3g_0g_2g_3$$

$$k_2 \equiv F_1F_2F_3c_0c_1c_3f_0f_1f_3-2E_1E_2E_3g_0g_1g_3$$

$$k_3 \equiv F_1F_2F_3c_0c_1c_2f_0f_1f_2-2E_1E_2E_3g_0g_1g_2$$

$$K_1 \equiv -G_2^2G_3^2+G_3^2G_1^2+G_1^2G_2^2$$

$$K_2 \equiv G_2^2G_3^2-G_3^2G_1^2+G_1^2G_2^2$$

$$K_3 \equiv G_2^2G_3^2+G_3^2G_1^2-G_1^2G_2^2.$$

The t's may now be written as

$$t_0 = 0 \qquad t_1 = \left(\frac{c-b}{c+b}\right)^{\frac{1}{2}}$$

$$t_2 = \frac{c+b}{a}.t_1 \qquad t_3 = \frac{c_0c_1}{c_2c_3}\,t_1$$

$$t_4 = \frac{2a^2E_1}{E_2E_3}\,t_2 \qquad t_5 = -\frac{f_0f_1}{f_2f_3}\,t_1$$

$$t_6 = \frac{F_0F_1}{F_2F_3}\,t_2 \qquad t_7 = -\frac{g_0g_1}{g_2g_3}\,t_1$$

$$t_8 = \frac{4a^2E_1E_2E_3G_1}{G_2G_3}.t_2 \qquad t_9 = \frac{c_0c_1h_0h_1}{c_2c_3h_2h_3}\,t_1$$

$$t_{10} = -\frac{H_0H_1}{H_2H_3}\,t_2 \qquad t_{11} = \frac{i_0i_1}{i_2i_3}\,t_1$$

$$t_{12} = \frac{I_0I_1}{I_2I_3}\,t_4 = \frac{2a^2c_0c_1c_2c_3E_1F_1F_2F_3I_1}{E_2E_3I_2I_3}.t_2$$

$$t_{13} = -\frac{j_0j_1}{j_2j_3}\,t_1 \qquad t_{14} = -\frac{J_0J_1}{J_2J_3}\,t_2$$

$$t_{15} = -\frac{c_0c_1f_0f_1k_0k_1}{c_2c_3f_2f_3k_2k_3}\,t_1 \qquad t_{16} = \frac{8a^2E_1E_2E_3G_1G_2G_3K_1}{K_2K_3}\,t_2$$

The porisms obtainable by methods sketched above must all occur in their rationalized forms. To obtain Chaundy's irrational forms, we must apply the inverse transformation of T_0 and denote it by T_0^{-1}. This obviously implies

$$a \to u, \quad b \to v, \quad c \to w$$

It would accordingly be worth while to work out in terms of R, r, d the results operating with T_0^{-1} on some of the functions defined above. Once these have been calculated, we could almost instantaneously pass on from rational to irrational forms or vice versa. Remembering that

$$2u^2 = R-d+r, \quad 2v^2 = R+d+r, \quad 2w^2 = 2R,$$

it follows that:

$$T_0^{-1}E_1 = -u^2+v^2+w^2 = R+d = \frac{1}{a}$$

$$T_0^{-1}E_2 = u^2-v^2+w^2 = R-d = \frac{1}{b}$$

$$T_0^{-1}E_3 = u^2+v^2-w^2 = r = \frac{1}{c}$$

$$T_0^{-1}F_1 = R^2-d^2+2rd = \frac{e_1}{abc}$$

$$T_0^{-1}F_2 = R^2-d^2-2rd = \frac{e_2}{abc}$$

$$T_0^{-1}F_3 = d^2-R^2+2Rr = \frac{e_3}{abc}$$

$$T_0^{-1}F_0 = R^2-d^2+2Rr = \frac{e_0}{abc}$$

$$T_0^{-1}G_1 = (R^2-d^2)^2+4r^2Rd = E_1(abc)^{-2}$$
$$T_0^{-1}G_2 = (R^2-d^2)^2-4r^2Rd = E_2(abc)^{-2}$$
$$T_0^{-1}G_3 = 2r^2(R^2+d^2)-(R^2-d^2)^2 = E_3(abc)^{-2}$$
$$T_0^{-1}H_1 = 2rd(R^2-d^2)(R^2-d^2+2rd)-(R^2-d^2)^3-8r^3R^2d = f_1(abc)^{-3}$$
$$T_0^{-1}H_2 = 2rd(R^2-d^2)(d^2-R^2+2rd)-(R^2-d^2)^3+8r^3R^2d = f_2(abc)^{-3}$$
$$T_0^{-1}H_3 = 2Rr(R^2-d^2)(R^2-d^2-2Rr)+(R^2-d^2)^3-8r^3Rd^2 = f_3(abc)^{-3}$$
$$T_0^{-1}H_0 = 2Rr(R^2-d^2)(R^2-d^2+2Rr)-(R^2-d^2)^3-8r^3Rd^2 = f_0(abc)^{-3}$$
$$T_0^{-1}I_1 = F_1(abc)^{-4}$$
$$T_0^{-1}I_2 = F_2(abc)^{-4}$$
$$T_0^{-1}I_3 = F_3(abc)^{-4}$$
$$T_0^{-1}I_0 = F_0(abc)^{-4}$$
$$T_0^{-1}J_1 = g_1(abc)^{-6}$$
$$T_0^{-1}J_2 = g_2(abc)^{-6}$$
$$T_0^{-1}J_3 = g_3(abc)^{-6}$$
$$T_0^{-1}J_0 = g_0(abc)^{-6}$$
$$T_0^{-1}K_1 = G_1(abc)^{-8}$$
$$T_0^{-1}K_2 = G_2(abc)^{-8}$$
$$T_0^{-1}K_3 = G_3(abc)^{-8}$$
$$= \{(R^2-d^2)^2-2r^2(R^2+d^2)\}^4-16r^4(R^2-d^2)^4\{(R^2+r^2-d^2)^2-4R^2r^2\}.$$

5

We are now fully equipped to proceed to the calculation of a number of porisms.

$C_2(8)$ is given by

$$t_1t_2 = 1 \quad \text{or} \quad a+b = c \quad \text{or} \quad (R+d)^{-1}+(R-d)^{-1} = r^{-1}$$

On applying T_0^{-1}, we get Chaundy's irrational form

$$\sqrt{(R-d+r)}+\sqrt{(R+d+r)} = \sqrt{(2R)}$$

On changing the sign of c, we obtain $C_1(8)$ in the form

$$a+b+c = 0 \quad \text{or} \quad e_0 = 0 \quad \text{or} \quad (R+d)^{-1}+(R-d)^{-1}+r^{-1} = 0.$$

$C_2(6)$ is given by

$$t_1 = t_2 \quad \text{or} \quad -a+b+c = 0 \quad \text{or} \quad e_1 = 0$$

and, therefore, $C_1(6)$ by

$$a-b+c = 0 \quad \text{or} \quad e_2 = 0.$$

We have already seen that $C_2(4)$ is given by $R = d$ and $C_1(4)$ by $R+d = 0$. $K(4)$ is given by $t_4^2 = 1$ or $a^2+b^2 = c^2$ or $E_3 = 0$, $C_1(8)$ is given by T_2 on $R+d = 0$ leading to $E_3 = 0$ and hence $C_2(8) \equiv E_1 = 0$.

Hence $C_2(5)$ is given by $t_2t_3 = 1$ which leads to $f_3 = 0$

Also $C_1(5) \equiv f_3 = 0$, $C_1(10) \equiv f_2 = 0$, $C_2(10) \equiv f_1 = 0$

$$T_0^{-1}f_0 \equiv u(-u^2+v^2+w^2)+v(u^2-v^2+w^2)+w(u^2+v^2-w^2)+2uvw = 0$$

which is Chaundy's irrational form for $C_1(5)$, *viz.*,

$$\frac{u}{a}+\frac{v}{b}+\frac{w}{c}+2uvw = 0$$

The various ways of writing $f_0 = 0$ are

$$(a+b)(b+c)(c+a) = a^3+b^3+c^3,$$

$$(a+b+c)^3 = 4(a^3+b^3+c^3),$$

$$(-a+b+c)(a-b+c)(a+b-c)+4abc = 0,$$

$$\begin{vmatrix} e_0 & e_3 & e_2 \\ e_3 & e_0 & e_1 \\ e_2 & e_1 & e_0 \end{vmatrix} = 0,$$

$$a(-a^2+b^2+c^2)+b(a^2-b^2+c^2)+c(a^2+b^2-c^2)+2abc = 0,$$

$$e_0^{-1}+e_1^{-1}+e_2^{-1}+e_3^{-1} = 0, \quad \text{etc.}$$

$K(6)$ is given by T_2 on $C_2(8)$ or T_2 on e_3 or $F_3 = 0$. T_2 on $C_1(8)$ gives $e_0e_1e_2e_3$ which gives nothing new. $K(6)$ may also be written

$$E_1^{-1}+E_2^{-1} = E_3^{-1}$$

$C_1(12)$ is given by T_2 on $C_1(6)$ or $F_2 = 0$. $C_2(12)$ is accordingly $F_1 = 0$. $C_2(7)$ is given by $t_3t_4 = 1$ giving $g_3 = 0$. Hence

$$C_1(7) \equiv g_0 = 0, \quad C_1(14) \equiv g_2 = 0 \quad \text{and} \quad C_2(14) \equiv g_1 = 0.$$

$T_0^{-1}g_0 = 0$ gives

$$\frac{a}{u}+\frac{b}{v}+\frac{c}{w}+\frac{1}{2uvw} = 0$$

which is Chaundy's irrational representative and which excites admiration when compared with the corresponding representative for a pentagon. It follows that the porism for a heptagon can be exhibited in each of the six rational forms corresponding to the six forms given above for a pentagon.

$K(8)$ is given by T_2 on $K(4)$ or $G_3 = 0$ or $E_1^{-2}+E_2^{-2} = E_3^{-2}$

$C_1(16)$ is given by T_2 on $C_1(8)$ or $G_3 = 0$ or $E_1^{-2}+E_3^{-2} = E_2^{-2}$

and

$C_2(16)$ is given by T_2 on $C_2(8)$ or $G_1 = 0$ or $E_2^{-2}+E_3^{-2} = E_1^{-2}$

Hence $C_1(9)$ may be derived from T_3 on $C_1(3)$ or $h_0 = 0$.

$$C_2(9) \equiv h_3 = 0, \quad C_1(18) \equiv h_2 = 0, \quad \text{and} \quad C_2(18) \equiv h_1 = 0$$

h_0 may also be exhibited in the form

$$\frac{a}{F_1}+\frac{b}{F_2}+\frac{c}{F_3} = 0,$$

which is changed by T_0^{-1} into Chaundy's irrational representative, *viz.*,

$$\frac{\sqrt{(R-d+r)}}{R^2-d^2+2rd}+\frac{\sqrt{(R+d+r)}}{R^2-d^2-2rd}+\frac{\sqrt{(2R)}}{d^2-R^2+2Rr} = 0$$

$K(10)$ may best be obtained by T_2 on $f_3 = 0$ which turns into

$$H_3 = 0 \quad \text{or} \quad \frac{G_1}{E_1}+\frac{G_2}{E_2}+\frac{G_3}{E_3}+2E_1E_2E_3 = 0.$$

Accordingly,

$$C_1(20) \equiv H_2 = 0, \qquad C_2(20) \equiv H_1 = 0.$$

T_2 on $f_0=0$ gives $H_0=0$ or $f_0f_1f_2f_3=0$ which is nothing new.

We come next to the undecagon, which is given by $t_5t_6=1$ or $f_0f_1e_1e_2e_3F_1(b-c) = af_2f_3F_2F_3$. This may be rewritten in the form $i_3=0$ or

$$\frac{aE_1}{F_1}+\frac{bE_2}{F_2}-\frac{cE_3}{F_3} = \frac{2abc}{F_0}$$

Hence $C_1(11)\equiv i_0=0$ or

$$\frac{aE_1}{F_1}+\frac{bE_2}{F_2}+\frac{cE_3}{F_3}+\frac{2abc}{F_0} = 0.$$

On applying T_0^{-1} we obtain the irrational representative

$$\frac{u}{ae_1}+\frac{v}{be_2}+\frac{w}{ce_3}+\frac{2uvw}{e_0} = 0,$$

or more explicitly

$$\frac{(R+d)\sqrt{(R+r-d)}}{R^2-d^2+2rd}+\frac{(R-d)\sqrt{(R+r+d)}}{R^2-d^2-2rd}+\frac{r\sqrt{(2R)}}{d^2-R^2+2Rr}+\frac{\sqrt{\{2R(R+r+d)(R+r-d)\}}}{R^2-d^2+2Rr} = 0$$

This may be compared with Chaundy's result. It is obvious that Chaundy's result contains two errors. In the denominators of the first two terms of the porism as

exhibited above, Chaundy's porism contains $+2Rd$ and $-2Rd$, whereas these ought to be $+2rd$ and $-2rd$ respectively. With this modification, Chaundy's result tallies completely with the one given here. That these represent actual errors in calculation and not merely misprints follows from an inspection of $C(22)$, $K(22)$ and $C(44)$, all of which are tainted by the same two errors in $C(11)$. Consequently, Chaundy's $K(44)$—which he considers unprintable—must contain the same errors, as also every other porism deduced by him from $C(11)$.

It follows that $C_1(22)\equiv i_2=0$ and $C_2(22)\equiv i_1=0$. I exhibit $C_1(22)$ in its irrational form so that comparison may be made with Chaundy's result

$$\frac{(R+d)\sqrt{(d+r-R)}}{d^2-R^2+2rR}+\frac{(d-R)\sqrt{(d+r+R)}}{d^2-R^2-2rR}+\frac{r\sqrt{(2d)}}{R^2-d^2+2rd}+\frac{\sqrt{\{2d(d+r+R)(d+r-R)\}}}{d^2-R^2+2dr} = 0.$$

The correct forms of $C(11)$ and $C(22)$, both rational and irrational, appear, I believe, for the first time in this paper.

The majority of the porisms which follow are believed to have been given here for the first time in their explicit forms. We proceed further. $K(12)$ would result from T_2 on $K(6)$ or

$$T_2F_2 = 0 \quad \text{or} \quad I_3 = 0 \quad \text{or} \quad G_1^{-1}+G_2^{-1} = G_3^{-1}.$$

Hence

$$C_1(24) \equiv I_2 = 0 \quad \text{and} \quad C_2(24) \equiv I_1 = 0.$$

The porism of a 13-agon is furnished by $t_6t_7 = 1$ leading to

$$(b-c)F_0F_1g_0g_1 = ag_2g_3F_2F_3 \quad \text{or} \quad j_3 = 0,$$

Hence

$$C_1(13) \equiv j_0 = 0, \quad C_1(26) \equiv j_2 = 0 \quad \text{and} \quad C_2(26) \equiv j_1 = 0$$

$j_0 = 0$ may also be written as

$$\frac{aG_1}{F_1}+\frac{bG_2}{F_2}+\frac{cG_3}{F_3}+\frac{4abcE_1E_2E_3}{F_0} = 0.$$

On applying the irrationalizing transformation, this becomes

$$\frac{uE_1}{e_1}+\frac{vE_2}{e_2}+\frac{wE_3}{e_3}+\frac{4uvwabc}{e_0} = 0$$

or more explicitly

$$\frac{\{(R^2-d^2)^2+4r^2Rd\}\sqrt{(R-d+r)}}{R^2-d^2+2rd}+\frac{\{(R^2-d^2)^2-4r^2Rd\}\sqrt{(R+d+r)}}{R^2-d^2-2rd}$$

$$+\frac{\{2r^2(R^2+d^2)-(R^2-d^2)^2\}\sqrt{(2R)}}{d^2-R^2+2Rr}+\frac{\sqrt{\{2R(R-d+r)(R+d+r)\}}2r(R^2-d^2)}{R^2-d^2+2Rr} = 0.$$

We next attack the 14-agon. $K(14)$ will be given by either $t_7^2=1$ or by T_2 on $C_2(7)$ or by $T_2g_3=0$ or $J_3=0$. Accordingly

$$C_1(28) \equiv J_2 = 0 \quad \text{and} \quad C_2(28) \equiv J_1 = 0$$

$K(14)$ may be exhibited in the form

$$\frac{E_1}{G_1}+\frac{E_2}{G_2}-\frac{E_3}{G_3} = \frac{1}{2E_1E_2E_3}.$$

The hurdle provided by a 15-agon is more troublesome. Chaundy just clears it, though not too gracefully, in the second attempt. The source of trouble springs from the two degenerate relatives, the triangle and the pentagon, who manage to creep in singly or together, no matter how we attack the problem. If we set $t_5t_7=1$, the triangle and the pentagon both creep in along with the 15-agon; if we set $t_5t_{10}=1$, the triangle alone is present and if we set $t_5t_6=1$, the pentagon keeps company with the 15-agon. If we apply T_5 on C_5, the pentagon is present and even if we knew the transformation T_5, T_5 on C_3 would bring in the undesirable triangle. One might have been tempted to give up the 15-agon as hopeless and to run away from the heavy algebra which blocks the way. But Chaundy's result gives rise to a hope that the algebra would simplify tremendously, if gracefully performed, and so we gather courage and proceed.

We set $t_5t_{10}=1$, which may be considered to result out of T_5 on $t_1t_2=1$. The result simplifies to

$$aH_2H_3 = (b-c)f_0^2/f_1^2H_1,$$

which may be rewritten as

$$aH_2H_3+bH_3H_1-cH_1H_2 = 0$$

The symmetric porism must, therefore, run

$$aH_2H_3+bH_3H_1+cH_1H_2 = 0,$$

a symmetric expression of degree 25 in a, b, c; the redundant factor $a+b+c$ still remains to be weeded out. Before we plunge into the algebra, we note that

$$\frac{a}{H_1}+\frac{b}{H_2}+\frac{c}{H_3} = 0,$$

could also have resulted out of T_5 on $a+b+c=0$, and accordingly T_5 must be given by

$$a \to \frac{a}{H_1}, \quad b \to \frac{b}{H_2}, \quad c \to \frac{c}{H_3}.$$

We have here stumbled across a gem, which we were not looking for. Besides providing us with a tool far more powerful than T_3, it gives us a clue to the general law of replicating p times when p is a prime exceeding 2. We have only to note that T_3 implies $c \to c/K(6)$ and that T_5 involves $c \to c/K(10)$ to conjecture that in general T_p should involve $c \to c/K(2p)$. We shall have occasion later on to test this conjecture for $p=7$, when we come to $C(21)$.

On trying to remove the factor $a+b+c$, we find

$$aH_2H_3+bH_3H_1+cH_1H_2 \equiv (a+b+c)(e_1f_1F_1e_2f_2F_2e_3f_3F_3-2E_1g_1E_2g_2E_3g_3)$$

Hence $C_3(15)$ is given by $k_6=0$. Also $C_2(15) \equiv k_5=0$, $C_1(30) \equiv k_2=0$, $C_2(30) \equiv k_1=0$. Now

$$\begin{aligned} k_6 &= e_1e_2e_3f_1f_2f_3F_1F_2F_3-2g_1g_2g_3E_1E_2E_3 \\ &\equiv G_1G_2G_3-2\sum ab(F_1F_2F_3G_3\overline{E_1+E_2}+H_3E_3\overline{G_1+G_2}) \\ &\equiv G_1G_2G_3-4\sum ab(c^2F_1F_2F_3G_3+H_3E_3E_1^2E_2^2). \end{aligned}$$

Hence $C_1(15)$ is given by

$$G_1G_2G_3 = 4abcF_1F_2F_3(aG_1+bG_2+cG_3)+4E_1E_2E_3(abE_1E_2H_3+bcE_2E_3H_1+caE_3E_1H_2)$$

Operating with T_0^{-1} leads to the irrational representative

$$\frac{E_1E_2E_3}{4uvwabc} = e_1e_2e_3(uE_1+vE_2+wE_3)+\frac{af_1}{u}+\frac{bf_2}{v}+\frac{cf_3}{w}.$$

or

$$K(16) \equiv T_2K(8) \equiv T_2G_3 \equiv K_3 = 0$$

Hence

$$G_1^{-2}+G_2^{-2} = G_3^{-2}.$$

$$C_1(32) \equiv K_2 = 0 \quad \text{and} \quad C_2(32) \equiv K_1 = 0.$$

The 17-agon yields the porism on setting $t_0t_8=1$. This gives

Hence

$$C_2(17) \equiv 4a(c-b)c_0e_1h_0h_1E_1E_2E_3G_1-e_2e_3h_2h_3G_2G_3 = 0.$$

$$C_1(17) \equiv 4a(c+b)e_2e_3h_2h_3E_1E_2E_3G_1+c_0e_1h_0h_1G_2G_3 = 0,$$

and

$$C_1(34) \equiv 4a(c-b)c_0e_1h_0h_1E_1E_2E_3G_1+e_2e_3h_2h_3G_2G_3 = 0,$$

$$C_2(34) \equiv 4a(c+b)e_2e_3h_2h_3E_1E_2E_3G_1-c_0e_1h_0h_1G_2G_3 = 0.$$

T_2 on $C_2(9)$ gives $K(18)$. Hence

$$K(18) \equiv (E_1I_1)^{-1}+(E_2I_2)^{-1}-(E_3I_3)^{-1} = 0,$$

and

$$C_1(36) \equiv (E_1I_1)^{-1}-(E_2I_2)^{-1}+(E_3I_3)^{-1} = 0,$$

$$C_2(36) \equiv -(E_1I_1)^{-1}+(E_2I_2)^{-1}+(E_3I_3)^{-1} = 0.$$

The next polygon is the 19-agon. This is given by $t_0t_{10}=1$ leading to

Hence

$$C_2(19) \equiv (b-c)c_0e_1h_0h_1F_0H_1+ae_2e_3h_2h_3H_2H_3 = 0,$$

$$C_1(19) \equiv (b+c)e_2e_3h_2h_3F_0H_1+ac_0e_1h_0h_1H_2H_3 = 0,$$

and

$$C_2(38) \equiv (b+c)e_2e_3h_2h_3F_0H_1-ac_0e_1h_0h_1H_2H_3 = 0,$$

$$C_1(38) \equiv (b-c)c_0e_1h_0h_1F_0H_1-ae_2e_3h_2h_3H_2H_3 = 0.$$

T_2 on $K(10)$ implies T_2 on $H_c=0$. Hence

$$K(20) \equiv \frac{K_1}{G_1}+\frac{K_2}{G_2}-\frac{K_3}{G_3}-2G_1G_2G_3 = 0,$$

$$C_1(40) \equiv \frac{K_1}{G_1}-\frac{K_2}{G_2}+\frac{K_3}{G_3}-2G_1G_2G_3 = 0,$$

$$C_2(40) \equiv -\frac{K_1}{G_1}+\frac{K_2}{G_2}+\frac{K_3}{G_3}-2G_1G_2G_3 = 0.$$

The 21-agon proves to be as troublesome as the 15-agon, and has to be attacked by similar methods.

$$T_7 \text{ on } l_2 l_1 = 1 \text{ gives } l_{24} l_7 = 1,$$

which simplifies to

$$aJ_2J_3 + (c-b)g_0^2 g_1^2 J_1 = 0$$

or

$$\frac{a}{J_1} + \frac{b}{J_2} - \frac{c}{J_3} = 0$$

This could have been deduced by applying T_7 on $a+b-c=0$ for a triangle. Hence T_7 is defined by

$$a \to \frac{a}{J_1}, \quad b \to \frac{b}{J_2}, \quad c \to \frac{c}{J_3},$$

which corroborates our conjecture regarding the general law of transformation for replicating a prime number of times.

As in the case of a 15-agon, the porism is tainted by the presence of an irrelevant factor $a+b-c$ corresponding to a triangle. Now

$$aJ_2J_3 + bJ_3J_1 + cJ_1J_2 \equiv (a+b+c)[-4E_1E_2E_3G_1G_2G_3f_1f_2f_3h_0 + F_1F_2F_3c_1c_2c_3g_1g_2g_3i_0]$$

Hence $C_1(21)$ is given by

$$4E_1E_2E_3G_1G_2G_3f_1f_2f_3h_0 - c_1c_2c_3E_1F_2F_3g_1g_2g_3i_0 = 0$$

$$\therefore \quad C_2(21) \equiv 4E_1E_2E_3G_1G_2G_3f_0f_1f_2h_3 - c_0c_1c_2F_1F_2F_3g_0g_1g_2i_3 = 0,$$

$$C_1(42) \equiv 4E_1E_2E_3G_1G_2G_3f_1f_2f_3h_2 - c_0c_1c_3F_1F_2F_3g_0g_1g_3i_2 = 0,$$

and

$$C_2(42) \equiv 4E_1E_2E_3G_1G_2G_3f_0f_2f_3h_1 - c_0c_2c_3F_1F_2F_3g_0g_2g_3i_1 = 0.$$

The next porism concerns a 22-agon. T_2 on $i_3 = 0$ gives $K(22)$ in the form

$$\frac{G_1}{E_1l_1} + \frac{G_2}{E_2l_2} - \frac{G_3}{E_3l_3} - \frac{2E_1E_2E_3}{F_0F_1F_2F_3} = 0$$

Hence

$$C_1(44) \equiv \frac{G_1}{E_1l_1} - \frac{G_2}{E_2l_2} + \frac{G_3}{E_3l_3} - \frac{2E_1E_2E_3}{F_0F_1F_2F_3} = 0$$

and

$$C_2(44) \equiv -\frac{G_1}{E_1l_1} + \frac{G_2}{E_2l_2} + \frac{G_3}{E_3l_3} - \frac{2E_1E_2E_3}{F_0F_1F_2F_3} = 0.$$

On setting $t_{11}t_{12} = 1$, we obtain $C_2(23)$ in the form

$$2a(c-b)E_1F_0F_1F_2F_3l_1i_0i_1 - E_2E_3l_2l_3i_2i_3 = 0$$

from which

$$C_1(23) \equiv 2a(b+c)F_0F_1F_2F_3E_1l_1i_2i_3 + E_2E_3l_2l_3i_0i_1 = 0,$$

$$C_1(46) \equiv 2a(b-c)F_0F_1F_2F_3E_1l_1i_0i_1 - E_2E_3l_2l_3i_2i_3 = 0,$$

and

$$C_2(46) \equiv -2a(b+c)F_0F_1F_2F_3E_1l_1i_2i_3 + E_2E_3l_2l_3i_0i_1 = 0.$$

$K(24) \equiv T_2K(12) \equiv l_2l_3 = 0$ giving

$$K_1^{-1} + K_2^{-1} - K_3^{-1} = 0$$

Hence

and

$$C_1(48) \equiv K_1^{-1} - K_2^{-1} + K_3^{-1} = 0$$

$$C_2(48) \equiv -K_1^{-1} + K_2^{-1} + K_3^{-1} = 0$$

$C_1(25)$ is obtained by T_5 on $C_1(5)$ leading immediately to

$$C_1(25) \equiv (c_0k_0)^{-1} + (c_1k_1)^{-1} + (c_2k_2)^{-1} + (c_3k_3)^{-1} = 0.$$

$$\because \quad C_2(25) \equiv (c_0k_0)^{-1} - (c_1k_1)^{-1} - (c_2k_2)^{-1} + (c_3k_3)^{-1} = 0.$$

and

$$C_1(50) \equiv (c_0k_0)^{-1} - (c_1k_1)^{-1} + (c_2k_2)^{-1} - (c_3k_3)^{-1} = 0,$$

$$C_2(50) \equiv (c_0k_0)^{-1} + (c_1k_1)^{-1} - (c_2k_2)^{-1} - (c_3k_3)^{-1} = 0.$$

We proceed no further; not because further progress has become more difficult, but because the porisms cannot present any new features of interest. The same forms are observed to repeat themselves. To pursue the matter further is merely a matter of algebra and patience.

To demonstrate the power of the tools in our hands, we work out $C_1(121)$. This will be given by T_{11} on $i_0 = 0$. To define T_{11}, we set

$$L_1 \equiv F_0F_1F_2F_3(-G_1E_2E_3I_2I_3 + G_2E_3E_1I_3I_1 + G_3E_1E_2I_1I_2) - 2E_1^2E_2^2E_3^2$$

$$L_2 \equiv F_0F_1F_2F_3(G_1E_2E_3I_2I_3 - G_2E_3E_1I_3I_1 + G_3E_1E_2I_1I_2) - 2E_1^2E_2^2E_3^2$$

and

$$L_3 \equiv F_0F_1F_2F_3(G_1E_2E_3I_2I_3 + G_2E_3E_1I_3I_1 - G_3E_1E_2I_1I_2) - 2E_1^2E_2^2E_3^2$$

Then $C_1(121)$ is given by

$$\sum \frac{a\left[-\frac{a^2}{L_1^2} + \frac{b^2}{L_2^2} + \frac{c^2}{L_3^2}\right]}{L_1\left[-\frac{3a^4}{L_1^4} + \frac{b^4}{L_2^4} + \frac{c^4}{L_3^4} - \frac{2b^2c^2}{L_2^2L_3^2} + \frac{2c^2a^2}{L_3^2L_1^2} + \frac{2a^2b^2}{L_1^2L_2^2}\right]} + \frac{2abc}{\left(\frac{a}{L_1} + \frac{b}{L_2} + \frac{c}{L_3}\right)\left(-\frac{a}{L_1} + \frac{b}{L_2} + \frac{c}{L_3}\right)\left(\frac{a}{L_1} - \frac{b}{L_2} + \frac{c}{L_3}\right)\left(\frac{a}{L_1} + \frac{b}{L_2} - \frac{c}{L_3}\right)} = 0$$

This is homogeneous and symmetric in a, b, c and of degree 1815. Its irrational representative can be worked out without much trouble, if desired. $C_2(121)$, $C_1(242)$ and $C_2(242)$ follow readily on proper sign adjustments.

If our ambition had been a little more unbridled, we would have aimed at $C_1(1681)$. The method would have been to calculate t_{20} by T_2 on t_{10} and t_{21} by T_3 on t_7. Thus $t_{21}t_{20}$ equated to 1 would yield $C_2(41)$, from which $C_1(41)$ could immediately be written down. Then T_2 on $C_2(41)$ would yield $K(82)$, from which $C_1(164)$, $C_2(164)$ would come out by proper sign adjustments. Then T_{41} is defined by

$$a \to \frac{a}{C_2(184)}, \quad b \to \frac{b}{C_1(184)}, \quad c \to \frac{c}{K(82)}$$

T_{41} on $C_1(41)$ yields $C_1(1681)$. But an actual working out would not be very profitable, although no insuperable difficulties crop up in the algebra. The final expression would be homogeneous and symmetrical in a, b, c and its degree would be 359,010. By such methods $K(2{\cdot}28^2)$ or $K(1058)$ and many similar porisms come out with great ease of calculation.

6

Considerations of completeness impel us to introduce elliptic functions at this conclusive stage. The treatment is in principle suggested by Clifford's work on two-two correspondences. Let

$$f(t, t') \equiv a(b-c)(1+t^2t'^2)-4bctt'+a(b+c)(t^2+t'^2) = 0$$

$$\therefore \quad dt\Big/\frac{\partial f}{\partial t'} + dt'\Big/\frac{\partial f}{\partial t} = 0$$

or

$$\frac{dt}{\sqrt{(1+2\lambda t^2+t^4)}} = \frac{dt'}{\sqrt{(1+2\lambda t'^2+t'^4)}}$$

where

$$\lambda = 1+\frac{2c^2(a^2-b^2)}{a^2(b^2-c^2)}$$

We assume $\lambda > 1$ and set

$$\lambda = \cosh\omega, \qquad x = \frac{te^{\omega/2}}{\sqrt{(1+t^2e^{\omega})}} \quad \text{and} \quad x' = \frac{t'e^{\omega/2}}{\sqrt{(1+t'^2e^{\omega})}}$$

This leads to

$$\int \frac{dx}{\sqrt{\{(1-x^2)(1-x^2k^2)\}}} = \int \frac{dx'}{\sqrt{\{(1-x'^2)(1-k^2x'^2)\}}}$$

where $k^2 = 1-e^{-2\omega}$, yielding

$$\mathrm{sn}^{-1}(x, k)-\mathrm{sn}^{-1}(x', k) = \text{constant}$$

Suppose $x' = 0$ gives $x = x_1$. Hence

$$\mathrm{sn}^{-1}(x, k)-\mathrm{sn}^{-1}(x', k) = \mathrm{sn}^{-1}(x_1, k).$$

Since $x_n = 0$,

$$\mathrm{sn}^{-1}(x_n, k) = n\,\mathrm{sn}^{-1}(x_1, k) = K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{(1-k^2\sin^2\theta)}}$$

as the condition of porism.

Hence

$$x_1 = \mathrm{sn}\left\{\frac{K}{n}, k\right\} = \frac{t_1e^{\omega/2}}{\sqrt{(1+t_1^2e^{\omega})}}$$

$$\therefore \quad \mathrm{tn}\left\{\frac{K}{n}\right\} = \left\{\frac{c-b}{c+b}\right\}^{\frac{1}{2}} \frac{\sqrt{(\lambda+1)}+\sqrt{(\lambda-1)}}{\sqrt{2}}$$

or

$$\mathrm{tn}\left\{\frac{K}{n}\right\} = \frac{c\sqrt{(b^2-a^2)}+b\sqrt{(c^2-a^2)}}{a(c+b)}.$$

Obviously, this cannot be the simplest condition for the porism of an n-agon in general, unless n be prime.

n = number of sides, d = degree of the rational porism in a, b, c

n	d	n	d	n	d	n	d	n	d	n	d
3	1	45	210	87	810	129	1848	171	3216	213	5040
4	2	46	261	88	900	130	2016	172	3306	214	5124
5	3	47	276	89	990	131	2145	173	3741	215	5341
6	4	48	256	90	880	132	1920	174	3360	216	5184
7	6	49	294	91	1008	133	2160	175	3906	217	5760
8	8	50	300	92	1056	134	2244	176	3840	218	5940
9	9	51	288	93	960	135	1944	177	3180	219	5928
10	12	52	336	94	1104	136	2304	178	3960	220	5760
11	15	53	351	95	1080	137	2816	179	4005	221	6048
12	16	54	324	96	1024	138	2112	180	3840	222	5472
13	21	55	360	97	1176	139	3145	181	4095	223	6216
14	24	56	384	98	1176	140	2904	182	4032	224	6144
15	24	57	360	99	1056	141	3268	183	4320	225	5400
16	32	58	420	100	1320	142	3520	184	4224	226	6384
17	36	59	435	101	1275	143	3520	185	4104	227	6441
18	36	60	384	102	1152	144	2904	186	3840	228	5760
19	48	61	465	103	1326	145	3920	187	4320	229	6555
20	48	62	480	104	1344	146	3864	188	4416	230	6336
21	48	63	432	105	1152	147	3552	189	3888	231	5760
22	60	64	512	106	1404	148	3696	190	4896	232	6720
23	66	65	504	107	1431	149	3775	191	4900	233	6720
24	64	66	480	108	1296	150	3400	192	4608	234	6048
25	75	67	561	109	1485	151	3850	193	4800	235	6624
26	84	68	576	110	1440	152	3840	194	4704	236	6960
27	81	69	528	111	1368	153	3592	195	4032	237	6240
28	90	70	576	112	1520	154	3888	196	4704	238	6912
29	105	71	630	113	1560	155	3880	197	4851	239	7140
30	96	72	576	114	1440	156	3888	198	4920	240	6144
31	120	73	666	115	1581	157	3981	199	4950	241	7260
32	120	74	681	116	1680	158	4120	200	4800	242	7260
33	120	75	690	117	1512	159	3888	201	4488	243	6561
34	144	76	720	118	1740	160	3072	202	5103	244	7140
35	144	77	720	119	1728	161	4368	203	5040	245	7056
36	144	78	672	120	1536	162	3960	204	4608	246	6720
37	171	79	780	121	1815	163	4021	205	5040	247	7500
38	180	80	768	122	1860	164	4300	206	5004	248	7560
39	168	81	729	123	1680	165	3880	207	4752	249	6888
40	192	82	840	124	1920	166	4414	208	5376	250	7000
41	210	83	861	125	1875	167	4486	209	5400	251	7875
42	192	84	768	126	1728	168	4072	210	4800	252	6912
43	231	85	864	127	2016	169	4840	211	5565	253	7920
44	240	86	924	128	2048	170	4480	212	5616	254	8064

7

All porisms are homogeneous in a, b, c. It is noticed that $C(3)$ is of degree 1, $C(5)$ of degree 3, $C(7)$ of degree 6 and in general $C(p)$ is of degree $\frac{1}{2}(p^2-1)$ for prime p. If n be a product of two distinct primes p and q, each different from 2, then the porisms corresponding to p and q intrude as irrelevant factors. Hence the chaste porism $C(pq)$ will be homogeneous in a, b, c and of degree $\frac{1}{2}[(p^2q^2-1)-(p^2-1)-(q^2-1)]$ or $\frac{1}{2}(p^2-1)(q^2-1)$. However, if n is a prime square, the degree of $C(p^2)$ is $\frac{1}{2}[(p^4-1)-(p^2-1)] = \frac{1}{2}p^2(p^2-1)$. Now since $C(p)$ is of degree $\frac{1}{2}(p^2-1)$, $K(2p)$ will be of degree $\frac{1}{2}(p^2-1)$, and hence the transformation T_p will multiply the degree of $C(p)$ by $\frac{1}{2}(p^2-1).2+1$ or by p^2. Hence T_p on C_p is of degree $p^2.\frac{1}{2}(p^2-1)$ or $\frac{1}{2}p^2(p^2-1)$. It follows that $C(p^2)$, given by T_p on $C(p)$, contains no irrelevant factors.

By similar considerations, it follows that $C(pqr)$, where $p, q, r > 2$ and prime, would be of degree

$$\frac{1}{2}[(p^2q^2r^2-1)-(p^2-1)-(q^2-1)-(r^2-1)-(p^2-1)(q^2-1)-(q^2-1)(r^2-1)$$
$$-(r^2-1)(p^2-1)] = \frac{1}{2}(p^2-1)(q^2-1)(r^2-1).$$

Now T_p on $C(q)$ is of degree $\frac{1}{2}p^2(q^2-1)$ and hence contains a factor of degree $\frac{1}{2}(q^2-1)$ which corresponds to $C(q)$. Hence

$$T_p \text{ on } C(q) \equiv C(q)\, C(pq).$$

Similarly

$$T_q \text{ on } C(p) \equiv C(p)\, C(pq).$$

We came across this phenomenon when discussing $C(15)$ and $C(21)$. But T_p on $C(pq)$ is of degree $\frac{1}{2}p^2(p^2-1)(q^2-1)$ and hence contains no irrelevant factors. In general, it is found that

$$T_p^{\alpha-1}\, T_q^{\beta-1}\, C(pqr) \equiv C(p^\alpha q^\beta r),$$

and contains no irrelevant factors.

As the transformation T_p multiplies the dimension by 4, the counting of dimensions for even porisms presents no problem whatsoever. The table given on page 103 gives the dimensions of porisms for an n-agon for $3 \leq n \leq 254$. An interesting fact, capable of a simple proof, emerges from the table. If the dimension of a porism is odd, the polygon has p^r sides, where p is prime and r a positive integer.

The full invariant forms for the porisms in terms of $\Delta, \Delta', \Theta, \Theta'$ may be obtained by the transformation

$$\frac{\Delta}{R^2} = \frac{\Theta}{2R^2+r^2-d^2} = \frac{\Theta'}{R^2+2r^2-d^2} = \frac{\Delta'}{r^2},$$

but like Chaundy, I am one of those who prefer to see the porisms exhibited in circular terms.

DEPARTMENT OF MATHEMATICS,

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