Poncelet Polygons and the Painlevé Equations

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Dedicated to M. S. Narasimhan and C. S. Seshadri
on the occasion of their 60th birthdays

1 Introduction

The celebrated theorem of Narasimhan and Seshadri [13] relating stable vector bundles on a curve to unitary representations of its fundamental group has been the model for an enormous range of recent results intertwining algebraic geometry and topology. The object which mediates between the two areas in all of these generalizations is the notion of a *connection*, and existence theorems for various types of connection provide the means of establishing the theorems. In one sense, the motivation for this paper is to pass beyond the existence and demand more explicitness. What do the connections look like? Can we write them down? This question is our point of departure. The novelty of our presentation here is that the answer involves a journey which takes us backwards in time over two hundred years from the proof of Narasimhan and Seshadri's theorem in 1965.

For simplicity, instead of considering stable bundles on curves of higher genus we consider the analogous case of parabolically stable bundles, in the sense of Mehta and Seshadri [11], on the complex projective line $\mathbf{C}\mathrm{P}^1$. Such a bundle consists of a vector bundle with a weighted flag structure at n marked points $a_1, \ldots, a_n$. The unitary connection that is associated with it is flat and has singularities at the points. In the generic case, the vector bundle itself is trivial, and the flat connection we are looking for can be written as a meromorphic $m \times m$ matrix-valued 1-form with a simple pole at each point a_i. The parabolic structure can easily be read off from the residues of the form. The other side of the equation is a representation of the fundamental group $\pi_1(\mathbf{C}\mathrm{P}^1 \backslash \{a_1, \ldots, a_n\})$ in $U(m)$, the holonomy of the connection, and this presents more problems. Such questions occupied the attention of Fuchs, Klein and others in the last century under the alternative name of monodromy of ordinary differential equations. Now if we fix the holonomy, and ask for the corresponding 1-form for each set of distinct points $\{a_1, \ldots, a_n\} \subset \mathbf{C}\mathrm{P}^1$, what in fact we are asking for is a solution of a differential equation, the so-called Schlesinger equation (1912) of isomonodromic deformation theory. To focus things even more, in the simple case where $m = 2$ and $n = 4$, an explicit form for the connection demands a knowledge of solutions to a single nonlinear second order differential equation. This equation, originally found in the context of isomonodromic deformations by R. Fuchs in 1907 [4], is nowadays called Painlevé's 6th equation

$$\begin{aligned} \frac{d^2y}{dx^2} &= 1/2\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\frac{dy}{dx} \\ &+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right) \end{aligned}$$

and in the words of Painlevé, the general solutions of this equation are "transcendantes essentiellement nouvelles". That, on the face of it, would seem to be the end of the quest for explicitness – we are faced with the insuperable obstacle of Painlevé transcendants.

Notwithstanding Painlevé's statement, for certain values of the constants $\alpha, \beta, \gamma, \delta$, there do exist solutions to the equation which can be written down, and even solutions that are *algebraic*. One property of any solution to the above equation is that $y(x)$ can only have branch points at $x = 0, 1, \infty$. This is essentially the "Painlevé property", that there are no movable singularities. If we find an algebraic solution, then this means we have an algebraic curve with a map to $\mathbf{C}\mathrm{P}^1$ with only three critical values. Such a curve is a modular curve, and, by a well-known theorem of Weil, is defined over $\bar{\mathbf{Q}}$. In this paper we shall construct solutions by considering the case when the holonomy group Γ of the connection is *finite*. In that case the solution $y(x)$ to the Painlevé equation is algebraic.

Our approach here is to consider, for a finite subgroup Γ of $SL(2, \mathbf{C})$, the quotient space $SL(2, \mathbf{C})/\Gamma$ and an equivariant compactification Z. Thus Z is a smooth projective threefold with an action of $SL(2, \mathbf{C})$ and a dense open orbit. The Maurer-Cartan form defines a flat connection on $SL(2, \mathbf{C})/\Gamma$ with holonomy Γ, which extends to a meromorphic connection on Z. The idea is then to look for rational curves in Z such that the induced connection is of the required form. By construction the holonomy is Γ, and if we can find enough curves to vary the cross-ratio of the singular points $a_1, \ldots, a_4$, then we have a solution to the Painlevé equation. The question of finding and classifying such equivariant compactifications has been addressed by Umemura and Mukai [12], but here we focus on one particular case. We take Γ to be the binary dihedral group $\tilde{D}_k \subset SU(2)$. This might seem very restrictive within the context of parabolically stable bundles, but behind it there hides a very rich seam of algebraic geometry which has its origins further back in history than Painlevé.

In the case of the dihedral group, the construction of a suitable compactification is due to Schwarzenberger [16], who constructed a family of rank 2 vector bundles V_k over $\mathbf{C}\mathrm{P}^2$. The threefold corresponding to the dihedral group D_k turns out to be the projectivized bundle $P(V_k)$. There are two types of relevant rational curves. Those which project to a line in $\mathbf{C}\mathrm{P}^2$ yield the solution $y = \sqrt{x}$ to the Painlevé equation with coefficients $(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/2k^2, 1/2 - 1/2k^2)$. Those which project to a conic lead naturally to another problem, and this one goes back at least to 1746 (see [3]). It is the problem of *Poncelet polygons*. We seek conics B and C in the plane such that there is a k-sided polygon inscribed in C and circumscribed about B. Interest in this problem is still widespread. Poncelet polygons occur in questions of stable bundles on projective spaces [14] and more recently in the work of Barth and Michel [1]. In fact, we can use their approach to find the modular curve giving the algebraic solution $y(x)$ of the Painlevé equation corresponding to $\Gamma = \tilde{D}_k$. This satisfies Painlevé's equation with coefficients $(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/8, 3/8)$. It is essentially Cayley's solution in 1853 of the Poncelet problem which allows us to go further and produce explicit solutions. It is a method which fits in well with the isomonodromic approach.

There are a number of reasons why this is a fruitful area of study. One of them concerns solutions of Painlevé equations in general and their relation to integrable systems, another is the connection with self-dual Einstein metrics as discussed in [6]. In the latter context, the threefolds constructed are essentially twistor spaces, and the rational curves twistor lines, but we shall not pursue this line of approach here. Perhaps the most intriguing challenge is to find *any* explicit solutions to an equation to which Painlevé's remark refers.

The structure of the paper is as follows. In Section 2 we consider singular connections and the isomonodromic deformation problem, and in Section 3 see how equivariant compactifications give solutions to the problem. In Section 4 we look at the way the dihedral group fits in with the problem of Poncelet polygons. Sections 5 and 6 discuss the actual solutions of the Painlevé equation, especially for small values of k. Only there can we see in full explicitness the connection which, in the context of the theorem of Narasimhan and Seshadri, relates the parabolic structure and the representation of the fundamental group, however restricted this example may be. In the final section we discuss the modular curve which describes the solutions so constructed.

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2 Singular connections

We introduce here the basic objects of our study – flat meromorphic connections with singularities of a specified type. For the most part we follow the exposition of Malgrange [10].

Definition 1 *Let Z be a complex manifold, Y a smooth hypersurface and E a holomorphic vector bundle over Z. Let ∇ be a flat holomorphic connection on E over $Z\backslash Y$ with connection form A in some local trivialization of E. Then on $U \subseteq Z$ we say that*

1. *∇ is meromorphic if A is meromorphic on U*
2. *∇ has a logarithmic singularity along Y if, in a local coordinate system $(z_1, \ldots, z_n)$ of Z, with Y given by $z_1 = 0$, A has the form*

$$A = A_1 \frac{dz_1}{z_1} + A_2 dz_2 + \ldots + A_n dz_n$$

where A_i is holomorphic on U.

One may easily check that the definition is independent of the choice of coordinates and local trivialization. The essential point about a logarithmic singularity is that the pole only occurs in the conormal direction to Y. In fact ∇ defines a holomorphic connection on E restricted to Y, with connection form

$$A_Y = \sum_{i=2}^{n} A_i(0, z_2, \ldots, z_n) dz_i.$$

If Z is 1-dimensional, then such a connection is just a meromorphic connection with simple poles. Flatness is automatic because the holomorphic curvature is a $(2, 0)$ form which is identically zero in one dimension. If we take $Z = \mathbf{C}\mathrm{P}^1$, $Y = \{a_1, \ldots, a_n, \infty\}$ and the bundle E to be trivial, then A is a matrix-valued meromorphic 1-form with simple poles at $z = a_1, \ldots, a_n, \infty$ and can thus be written as

$$A = \sum_{i=1}^{n} \frac{A_i dz}{z - a_i}$$

The holonomy of a flat connection on $Z\backslash Y$ is obtained by parallel translation around closed paths and defines, after fixing a base point b, a representation of the fundamental group

$$\rho : \pi_1(Z\backslash Y) \to GL(m, \mathbf{C}).$$

In one dimension, the holonomy may also be considered as the effect of analytic continuation of solutions to the system of ordinary differential equations

$$\frac{df}{dz} + \sum_{i=1}^{n} \frac{A_i f}{z - a_i} = 0$$

around closed paths through b. As such, one often uses the classical term *monodromy* rather than the differential geometric *holonomy*. Changing the basepoint to b' effects an overall conjugation (by the holonomy along a path from b to b') of the holonomy representation.

For the punctured projective line above, we obtain a representation of the group $\pi_1(S^2\backslash\{a_1, \ldots, a_n, \infty\})$. This is a free group on n generators, which can be taken as simple loops γ_i from b passing once around a_i. Moving b close to a_i, it is easy to see that $\rho(\gamma_i)$ is conjugate to

$$\exp(-2\pi i A_i).$$

There is also a singularity of A at infinity with residue A_∞. Since the sum of the residues of a differential is zero, we must have

$$A_\infty = -\sum_{i=1}^{n} A_i$$

and so $\rho(\gamma_\infty)$ is conjugate also to $\exp(-2\pi i A_\infty)$. In the fundamental group itself $\gamma_1\gamma_2 \ldots \gamma_n\gamma_\infty = 1$ so that in the holonomy representation

$$\rho(\gamma_1)\rho(\gamma_2)\ldots\rho(\gamma_\infty) = 1. \tag{1}$$

Thus the conjugacy classes of the residues A_i of the connection determine the conjugacy classes of $\rho(\gamma_i)$, and these must also satisfy (1).

This is partial information about the holonomy representation. However, the full holonomy group depends on the position of the poles a_i. The problem of particular interest to us here is the *isomonodromic deformation problem* – to determine the dependence of A_i on $a_1, \ldots, a_n$ in order that the holonomy representation should remain the same up to conjugation. All we have seen so far is that the conjugacy class of $\exp(2\pi i A_i)$ should remain constant.

One way of approaching the isomonodromic deformation problem, due to Malgrange, is via a universal deformation space. Let X_n denote the space of ordered distinct points $(a_1, \ldots, a_n) \in \mathbf{C}$, and $\tilde{X}_n$ its universal covering. It is well-known that this is a contractible space – the classifying space for the braid group on n strands. Now consider the divisor

$$Y_m = \{(z, a_1, \ldots, a_n) \in \mathbf{CP}^1 \times X_n : z \neq a_m\}$$

and let $\tilde{Y}_m$ be its inverse image in $\mathbf{C}\mathrm{P}^1 \times \tilde{X}_n$. Furthermore, define $\tilde{Y}_\infty = \{(\infty, x) : x \in \tilde{X}_n\}$.

The projection onto the second factor $p : \mathbf{C}\mathrm{P}^1 \times \tilde{X}_n \backslash \{\tilde{Y}_1 \cup \ldots \cup \tilde{Y}_n \cup \tilde{Y}_\infty\} \to \tilde{X}_n$ is a fibration with fibre $\mathbf{C}\mathrm{P}^1 \backslash \{a_1, \ldots, a_n, \infty\}$, and the contractibility of $\tilde{X}_n$ implies from the exact homotopy sequence that the inclusion i of a fibre induces an isomorphism of fundamental groups

$$\pi_1(\mathbf{C}\mathrm{P}^1 \backslash \{a_1^0, \ldots, a_n^0, \infty\}) \cong \pi_1(\mathbf{C}\mathrm{P}^1 \times \tilde{X}_n \backslash \{\tilde{Y}_1 \cup \ldots \cup \tilde{Y}_n \cup \tilde{Y}_\infty\})$$

Thus a flat connection on $\mathbf{C}\mathrm{P}^1 \backslash \{a_1^0, \ldots, a_n^0, \infty\}$ extends to a flat connection with the same holonomy on $\mathbf{C}\mathrm{P}^1 \times \tilde{X}_n \backslash \{\tilde{Y}_1 \cup \ldots \cup \tilde{Y}_n \cup \tilde{Y}_\infty\}$. Malgrange's theorem asserts that this flat connection has logarithmic singularities along $\tilde{Y}_m$ and $\tilde{Y}_\infty$.

More precisely,

Theorem 1 (Malgrange [10]) *Let ∇^0 be a flat holomorphic connection on the vector bundle E^0 over $\mathbf{C}\mathrm{P}^1 \backslash \{a_1^0, \ldots, a_n^0, \infty\}$, with logarithmic singularities at $a_1^0, \ldots, a_n^0$. Then there exists a holomorphic vector bundle E on $\mathbf{C}\mathrm{P}^1 \times \tilde{X}_n$ with a flat connection ∇ with logarithmic singularities at $\tilde{Y}_1, \ldots, \tilde{Y}_n, \tilde{Y}_\infty$ and an isomorphism $j : i^*(E, \nabla) \to (E^0, \nabla^0)$. Furthermore, (E, ∇, j) is unique up to isomorphism.*

Now suppose that E^0 is holomorphically trivial. The vector bundle E will not necessarily be trivial on all fibres of the projection p, but for a dense open set $U \subseteq \tilde{X}_n$ it will be. Choose a basis $e_1^0, e_2^0, \ldots, e_m^0$ of the fibre of E^0 at $z = \infty$. Now since ∇ has a logarithmic singularity on $\tilde{Y}_\infty$, it induces a flat connection there, and since $\tilde{Y}_\infty \cong \tilde{X}_n$ is simply connected, by parallel translation we can unambiguously extend $e_1^0, e_2^0, \ldots, e_m^0$ to a trivialization of E over $\tilde{Y}_\infty$. Then since E is holomorphically trivial on each fibre over U, we can uniquely extend $e_1^0, e_2^0, \ldots, e_m^0$ along the fibres to obtain a trivialization $e_1, \ldots, e_m$ of E on $\mathbf{C}\mathrm{P}^1 \times U$. It is easy to see that, relative to this trivialization, the connection form of ∇ can be written

$$A = \sum_{i=1}^{n} A_i \frac{dz - da_i}{z - a_i} \tag{2}$$

where A_i is a holomorphic function of $a_1, \ldots, a_n$.

The flatness of the connection can then be expressed as:

$$dA_i + \sum_{j \neq i} [A_i, A_j] \frac{da_i - da_j}{a_i - a_j} = 0$$

which is known as *Schlesinger's equation* [15].

The gauge freedom in this equation involves only the choice of the initial basis $e_1^0, e_2^0, \ldots, e_m^0$ and consists therefore of conjugation of the A_i by a constant matrix.

The case which interests us here is where the holonomy lies in $SL(2,\mathbf{C})$, (so that the A_i are trace-free 2×2 matrices), and where there are 3 marked points a_1, a_2, a_3 which, together with $z=\infty$, are the singular points of the connection. By a projective transformation we can make these points $0, 1, x$. Then

$$A(z) = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-x}$$

and Schlesinger's equation becomes:

$$\begin{aligned} \frac{dA_1}{dx} &= \frac{[A_3, A_1]}{x} \\ \frac{dA_2}{dx} &= \frac{[A_3, A_2]}{x-1} \\ \frac{dA_3}{dx} &= \frac{-[A_3, A_1]}{x} - \frac{[A_3, A_2]}{x-1} \end{aligned} \tag{3}$$

where the last equation is equivalent to

$$A_1 + A_2 + A_3 = -A_\infty = \text{const.}$$

The relationship with the Painlevé equation can best be seen by following [8]. Each entry of the matrix $A_{ij}(z)$ is of the form $q(z)/z(z-1)(z-x)$ for some quadratic polynomial q. Suppose that A_∞ is diagonalizable, and choose a basis such that

$$A_\infty = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

then A_{12} can be written

$$A_{12}(z) = \frac{k(z-y)}{z(z-1)(z-x)} \tag{4}$$

for some $y \in \mathbf{C}\mathrm{P}^1\backslash\{0, 1, x, \infty\}$. If the $A_i(x)$ satisfy (3), then the function $y(x)$ satisfies the Painlevé equation

$$\begin{aligned} \frac{d^2y}{dx^2} &= 1/2\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\frac{dy}{dx} \\ &+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right) \end{aligned} \tag{5}$$

where

$$\begin{aligned} \alpha &= (2\lambda - 1)^2/2 \\ \beta &= 2\det A_1 \\ \gamma &= -2\det A_2 \\ \delta &= (1 + 4\det A_3)/2 \end{aligned} \tag{6}$$

For the formulae which reconstruct the connection from $y(x)$ we refer to [8], but essentially the entries of the A_i are rational functions of x, y and dy/dx. For our purposes it is useful to note the geometrical form of the definition of $y(x)$ given by (4):

Proposition 1 *The solution $y(x)$ to the Painlevé equation corresponding to an isomonodromic deformation $A(z)$ is the point $y \in \mathbf{C}\mathrm{P}^1 \backslash \{0, 1, x, \infty\}$ at which $A(y)$ and A_∞ have a common eigenvector.*

Note that strictly speaking there are two Painlevé equations (with $\alpha = (\pm 2\lambda - 1)^2/2$) corresponding to the values of y with this property.

3 Equivariant compactifications

Consider the three-dimensional complex Lie group $SL(2, \mathbf{C})$ and the Lie algebra-valued 1-form

$$A = -(dg)g^{-1}.$$

The form A is the connection form for a trivial connection on the trivial bundle. It simply relates the trivializations of the principal frame bundle by left and right translation.

Now let Γ be a finite subgroup of $SL(2, \mathbf{C})$. Then $SL(2, \mathbf{C})/\Gamma$ is a noncompact complex manifold and since A is invariant under right translation, it descends to this quotient. Thus, on $SL(2, \mathbf{C})/\Gamma$, A defines a flat connection on the trivial rank 2 vector bundle. Its holonomy is tautologically Γ.

In this section, we shall consider an equivariant compactification of $SL(2, \mathbf{C})/\Gamma$, that is to say, a compact complex manifold Z on which $SL(2, \mathbf{C})$ acts with a dense open orbit with stabilizer conjugate to Γ. Let Z be such a compactification, then the action of the group embeds the Lie algebra $\underline{g}$ in the space of holomorphic vector fields on Z. Equivalently, we have a vector bundle homomorphism

$$\alpha : Z \times \underline{g} \to TZ$$

which is generically an isomorphism. It fails to be an isomorphism on the union of the lower dimensional orbits of $SL(2, \mathbf{C})$, and this is where $\wedge^3 \alpha = 0$. But $\wedge^3 \alpha \in H^0(Z, \mathrm{Hom}(\wedge^3 \underline{g}, \wedge^3 T)) \cong H^0(Z, K^{-1})$ is a section of the anticanonical bundle, so the union of the orbits of dimension less than three form an *anticanonical divisor* Y, which may of course have several components or be singular.

On the open orbit $Z \backslash Y \cong SL(2, \mathbf{C})/\Gamma$, the action is equivalent to left multiplication, and the connection A above is given by

$$A = \alpha^{-1} : TZ \to Z \times \underline{g}.$$

It is clearly meromorphic on Z, but more is true:

Proposition 2 *If $\wedge^3\alpha$ vanishes non-degenerately on the divisor Y, then the connection $A = \alpha^{-1}$ has a logarithmic singularity along Y.*

This is a local statement, and so it can always be applied to the smooth part of Y even if there are singular points.

Proof: In local coordinates, α is represented by a holomorphic function $B(z)$ with values in the space of 3×3 matrices. The divisor Y is then the zero set of $\det B$. If $\det B$ has a non-degenerate zero at $p \in Y$, then its null-space is one-dimensional at p, so the kernel of α, the Lie algebra of the stabilizer of p, is one-dimensional. Thus the $SL(2, \mathbf{C})$ orbit through p is two-dimensional, and so Y is the orbit.

Now for any square matrix B, let $B^\vee$ denote the transpose of the matrix of cofactors. Then it is well-known that

$$BB^\vee = (\det B)I$$

Hence in local coordinates

$$A = \alpha^{-1} = \frac{B^\vee}{\det B}$$

and so A has a simple pole along Y. From Definition 1, we need to show that the residue is in the conormal direction. For this consider the invariant description of $B^\vee$. We have on Z

$$\wedge^2\alpha : \wedge^2\underline{g} \to \wedge^2 T$$

and using the identifications $\wedge^2\underline{g} \cong \underline{g}^*$ and $\wedge^2 T \cong T^* \otimes \wedge^3 T$, $B^\vee$ represents the dual map of $\wedge^2\alpha$:

$$(\wedge^2\alpha)^* : T \to \underline{g} \otimes \wedge^3 T.$$

Now the image of α at p is the tangent space to the orbit Y at p by the definition of α. Thus the image of $\wedge^2\alpha$ is $\wedge^2 TY_p$ which means that $(\wedge^2\alpha)^*$ annihilates TY, which is the required result.

Note that the kernel of $\wedge^2\alpha$ is the set of two-vectors $v \wedge w$ where $w \in \underline{g}$ and v is in the Lie algebra of the stabilizer of p. Thus the residue at p of the connection A lies in the Lie algebra of the stabilizer.

Now suppose that P is a rational curve in Z which meets Y transversally at four points. Then the restriction of A to P defines a connection with logarithmic poles at the points and, from the map of fundamental groups

$$\pi_1(P \backslash \{a_1, \ldots, a_4\}) \to \pi_1(Z \backslash Y) \to \Gamma \to SL(2, \mathbf{C}),$$

its holonomy is contained in Γ. A deformation of P will define a nearby curve in the same homotopy class and hence the induced connection will have the same holonomy. To obtain isomonodromic deformations, we therefore need to study the deformation theory of such curves.

Proposition 3 *Let $P \subset Z$ be a rational curve meeting Y transversally at four points. Then P belongs to a smooth four-parameter family of rational curves on which the cross-ratio of the points is a nonconstant function.*

Proof: The proof is standard Kodaira-Spencer deformation theory. By hypothesis P meets the anticanonical divisor Y in four points, so the degree of K_Z on P is -4. Hence, if N is the normal bundle of $P \cong \mathbf{C}\mathrm{P}^1$,

$$\deg N = -\deg K_Z + \deg K_P = 2$$

and so

$$N \cong \mathcal{O}(m) \oplus \mathcal{O}(2-m)$$

for some integer m. However, since C is transversal to the 2-dimensional orbit Y of $SL(2,\mathbf{C})$, the map α always maps *onto* the normal bundle to C. We therefore have a surjective homomorphism of holomorphic vector bundles

$$\beta : \mathcal{O} \otimes \underline{g} \to N$$

and this implies that $0 \leq m \leq 2$. As a consequence, $H^1(P,N) = 0$ and $H^0(P,N)$ is four-dimensional, so the existence of a smooth family follows from Kodaira [9].

Since β is surjective, its kernel is a line bundle of degree $-\deg N = -2$, so we have an exact sequence of sheaves:

$$\mathcal{O}(-2) \to \mathcal{O} \otimes \underline{g} \to N.$$

Under α, the kernel maps isomorphically to the sheaf of sections of the tangent bundle TP which vanish at the four points $P \cap Y$. From the long exact cohomology sequence we have

$$0 \to \underline{g} \to H^0(P,N) \xrightarrow{\delta} H^1(P,\mathcal{O}(-2)) \to 0$$

and since $H^0(P,N)$ is 4-dimensional and $\underline{g}$ is 3-dimensional, the map δ is surjective. But $\alpha\delta$ is the Kodaira-Spencer map for deformations of the four points on P, so since it is non-trivial, the cross-ratio is non-constant.

Example

As the reader may realize, the situation here is very similar to the study of twistor spaces and twistor lines, and indeed there is a differential geometric context for this (see [6], [7]). This is not the agenda for this paper, but it is a useful example to see the standard twistor space – $\mathbf{C}\mathrm{P}^3$ and the straight lines in it – within the current context.

Let V be the 4-dimensional space of cubic polynomials

$$p(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3$$

and consider V as a representation space of $SL(2, \mathbf{C})$ under the action

$$p(z) \mapsto p\left(\frac{az+b}{cz+d}\right)(cz+d)^3.$$

This is the unique (up to isomorphism) 4-dimensional irreducible representation of $SL(2, \mathbf{C})$. Then $Z = P(V) = \mathbf{C}\mathrm{P}^3$ is a compact threefold with an action of $SL(2, \mathbf{C})$ and moreover the open dense set of cubics with distinct roots is an orbit. This follows since given any two triples of distinct ordered points in $\mathbf{C}\mathrm{P}^1$, there is a unique element of $PSL(2, \mathbf{C})$ which takes one to the other. However, the cubic polynomial determines an *unordered* triple of roots, and hence the stabilizer in $PSL(2, \mathbf{C})$ is the symmetric group S_3. Thinking of this as the symmetries of an equilateral triangle, the holonomy group $\Gamma \subset SL(2, \mathbf{C})$ of the connection $A = \alpha^{-1}$ is the binary dihedral group $\tilde{D}_3$. The lower-dimensional orbits consist firstly of the cubics with one repeated root, which is 2-dimensional, and those with a triple root, which constitute a rational normal curve in $\mathbf{C}\mathrm{P}^3$. Together they form the discriminant divisor Y, the anticanonical divisor discussed above.

A generic line in $\mathbf{C}\mathrm{P}^3$, generated by polynomials $p(z), q(z)$ meets Y at those values of t for which the discriminant of $tp(z) + q(z)$ vanishes, i.e. where

$$\begin{aligned} tp(z) + q(z) &= 0 \\ tp'(z) + q'(z) &= 0 \end{aligned}$$

have a common root. This occurs for $t = -q(\alpha)/p(\alpha)$ where α is a root of the quartic equation

$$p'(z)q(z) - p(z)q'(z) = 0$$

and so the line meets Y in four generically distinct points. Thus the 4-parameter family of lines in $\mathbf{C}\mathrm{P}^3$ furnish an example of the above proposition.

As we remarked above, this is an example of an isomonodromic deformation, as would be any family of curves P in Proposition 3. It yields a solution of the Painlevé equation either by applying the arguments of Theorem 1 to the connection with logarithmic singularities on Z, or appealing to the universality of Malgrange's construction. We shall not derive the solution of the Painlevé equation here from $\mathbf{C}\mathrm{P}^3$, since it will appear via a different compactification in the context of Poncelet polygons. There we shall also see how a straight line in $\mathbf{C}\mathrm{P}^3$ defines a pair of conics with the Poncelet property for triangles.

4 Poncelet polygons and projective bundles

In this section we shall study a particular class of equivariant compactifications, originally due to Schwarzenberger [16]. Consider the complex surface $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$ and the holomorphic involution σ which interchanges the two factors. The quotient space is $\mathbf{C}\mathrm{P}^2$. A profitable way of viewing this is as the map which assigns to a pair of complex numbers the coefficients of the quadratic polynomial which has them as roots. In affine coordinates we have the quotient map

$$\begin{aligned} \pi : \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1 &\rightarrow \mathbf{C}\mathrm{P}^2 \\ (w, z) &\mapsto (-(w+z), wz). \end{aligned}$$

From this it is clear that π is a double covering branched over the image of the diagonal, which is the conic $B \subset \mathbf{C}\mathrm{P}^2$ with equation $4y = x^2$. Moreover, the line $\{a\} \times \mathbf{C}\mathrm{P}^1 \subset \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$ maps to a line in $\mathbf{C}\mathrm{P}^2$ which meets B at the single point $\pi(a, a)$. The images of the two lines $\{a\} \times \mathbf{C}\mathrm{P}^1$ and $\mathbf{C}\mathrm{P}^1 \times \{b\}$ are therefore the two tangents to the conic B from the point $\pi(a, b)$.

Now let $\mathcal{O}(k, l)$ denote the unique holomorphic line bundle of bidegree (k, l) on $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$, and define the direct image sheaf $\pi_* \mathcal{O}(k, 0)$ on $\mathbf{C}\mathrm{P}^2$. This is a locally free sheaf, a rank 2 vector bundle V_k, and we may form the projective bundle $P(V_k)$, a complex 3-manifold which fibres over $\mathbf{C}\mathrm{P}^2$

$$p : P(V_k) \rightarrow \mathbf{C}\mathrm{P}^2$$

with fibres $\mathbf{C}\mathrm{P}^1$.

Clearly the diagonal action of $SL(2, \mathbf{C})$ on $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$ induces an action on $P(V_k)$. Take a point $z \in P(V_k)$ and consider its stabilizer. If $p(z) \in \mathbf{C}\mathrm{P}^2 \backslash B$, then

$p(z) = \pi(a,b)$ where $a \neq b$. Consider the projective bundle pulled back to $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$. The point (a,b) is off the diagonal in $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$, so the fibre of $P(V_k) = P(\pi_* \mathcal{O}(k,0))$ is

$$P(\mathcal{O}(k,0)_a \oplus \mathcal{O}(k,0)_b). \tag{7}$$

The stabilizer of (a,b) in $SL(2,\mathbf{C})$ is one-dimensional, and acts on $(u,v) \in \mathcal{O}(k,0)_a \oplus \mathcal{O}(k,0)_b$ as

$$(u,v) \mapsto (\lambda^k u, \lambda^{-k} v).$$

Thus, as long as $u \neq 0$ or $v \neq 0$, the stabilizer of the point represented by (u,v) in the fibre is finite. Thus the generic orbit is three-dimensional.

We have implicitly just defined the divisor Y of lower-dimensional orbits, but to be more precise, we have the inverse image of the branch locus

$$D_1 = \pi^{-1}(B)$$

as one component. The other arises from the direct image construction as follows.

Recall that by definition of the direct image, for any open set $U \subseteq \mathbf{C}\mathrm{P}^2$,

$$H^0(U, V_k) \cong H^0(\pi^{-1}(U), \mathcal{O}(k,0))$$

so that there is an evaluation map

$$\mathrm{ev} : H^0(\pi^{-1}(U), \pi^* V_k) \to H^0(\pi^{-1}(U), \mathcal{O}(k,0)).$$

The kernel of this defines a distinguished line sub-bundle of $\pi^*(V_k)$ and thus a section of the pulled back projective bundle $P(V_k)$. This copy of $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$ in $P(V_k)$ is a divisor D_2.

Both divisors are components of the anticanonical divisor Y, and it remains to check the multiplicity. Now let U be the divisor class of the tautological line bundle over the projective bundle $P(V_k)$. The divisor D_2 is a section of $P(V_k)$ pulled back to $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$, and from its definition it is in the divisor class $p^*(-U) + \mathcal{O}(k,0)$. Thus in $P(V_k)$,

$$D_2 \sim -2U + kH \tag{8}$$

where H is the divisor class of the pull-back by π of the hyperplane bundle on $\mathbf{C}\mathrm{P}^2$. Clearly, since B is a conic,

$$D_1 \sim 2H. \tag{9}$$

Now from Grothendieck-Riemann-Roch applied to the projection π, we find $c_1(V_k) = (k-1)H$, from which it is easy to see that the canonical divisor class is

$$K \sim 2U - (k+2)H$$

so since $-K \sim -2U + (k+2)H \sim D_1 + D_2$, the multiplicity is 1 for each divisor, and we can take $Z = P(V_k)$ as an example of an equivariant compactification to which Proposition 2 applies.

The stabilizer of a point in $Z \backslash Y$ is in this case the binary dihedral group $\tilde{D}_k$, which is the inverse image in $SU(2)$ of the group of symmetries in $SO(3) \cong SU(2)/\pm 1$ of a regular plane polygon with k sides. Although this can be seen quite easily from the above description of the action, there is a direct way of viewing $Z \backslash Y = P(V_k) \backslash D_1 \cup D_2$ as the $SL(2, \mathbf{C})$ orbit of a plane polygon.

Note that a polygon centred on $0 \in \mathbf{C}^3$ is described by a non-null axis orthogonal to the plane of the polygon, and by k (if k is odd) or $k/2$ (if k is even) equally spaced axes through the origin in that plane. Now, given a point $z \in P(V_k) \backslash D_1 \cup D_2$, its projection $p(z) = x \in P(\mathbf{C}^3) \backslash B$ is a non-null direction in $\mathbf{C}^3$ which we take to be the axis. To find the other axes we use two facts:

- The map $s \mapsto s^k$ from $\mathcal{O}(1,0)$ to $\mathcal{O}(k,0)$ defines a rational map $m_k : P(V_1) \to P(V_k)$ of degree k.
- The projective bundle $P(V_2)$ is the projectivized tangent bundle $P(T)$ of $\mathbf{C}\mathrm{P}^2$.

The first fact is a direct consequence of the definition of the direct image sheaf:

$$H^0(U, V_k) \cong H^0(\pi^{-1}(U), \mathcal{O}(k,0))$$

for any open set $U \subseteq \mathbf{C}\mathrm{P}^2$. The second can be found in [16].

Given these two facts, consider the set of points

$$m_2(m_k^{-1}(z)) \subset P(V_2).$$

Depending on the parity of k this consists of k or $k/2$ points in $P(T)$ all of which project to $x \in \mathbf{C}\mathrm{P}^2$. In other words they are lines through x or, using the polarity with respect to the conic B, points on the polar line of x. Reverting to linear algebra, these are axes in the plane orthogonal to x.

We now need to apply Proposition 3 to this particular set of examples – to find rational curves which meet the divisor $Y = D_1 + D_2$ transversally in four points. Now if P is such a curve, then the intersection number $P.D_1 \leq 4$ so $p(P) = C$ is a plane curve of degree d which meets the branch conic B in $2d \leq 4$ points, hence $d = 1$ or 2. We consider the case $d = 2$ first. The curve C is a conic in $\mathbf{C}\mathrm{P}^2$. The set of all conics forms a 5-parameter family, and we want to determine the 4-parameter family of conics which lift to $P(V_k)$.

Theorem 2 *A conic $C \subset \mathbf{C}\mathrm{P}^2$ meeting B transversally lifts to $P(V_k)$ if and only if there exists a k-sided polygon inscribed in C and circumscribed about B.*

Proof: A lifting of C is a section of $P(V_k)$ over C, or equivalently a line subbundle $M \subset V_k$ over C. Since C is a conic, the hyperplane bundle H is of degree 2 on C, so we can write $M \cong H^{n/2}$ for some integer n. The inclusion $M \subset V_k$ thus defines a holomorphic section s of the vector bundle $V_k \otimes H^{-n/2}$ over C. But V_k is the direct image sheaf of $\mathcal{O}(k,0)$, so we have an isomorphism

$$H^0(C, V_k \otimes H^{-n/2}) \cong H^0(\tilde{C}, \mathcal{O}(k,0) \otimes \pi^*(H^{-n/2}))$$

where $\tilde{C} = \pi^{-1}(C) \subset \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$ is the double covering of the conic C branched over its points of intersection with B. But it is easy to see that $\pi^*(H) \cong \mathcal{O}(1,1)$, so on $\tilde{C}$ we have a holomorphic section $\tilde{s}$ of $\mathcal{O}(k - n/2, -n/2)$.

We have more, though, for since the intersection number $-K.P = (D_1+D_2).P = 4$ and $D_1.P = B.C = 4$, P lies in $P(V_k)\backslash D_2$, where D_2 was given as the kernel of the evaluation map. If the section $\tilde{s}$ vanishes anywhere, then the section of $P(V_k)$ will certainly meet D_2, thus $\tilde{s}$ is everywhere non-vanishing and $\mathcal{O}(k - n/2, -n/2)$ is the trivial bundle. In particular, its degree is zero on $\tilde{C}$. Now C is a conic, so $\tilde{C}$ is the divisor of a section of $\pi^*(H^2) \cong \mathcal{O}(2,2)$ and so the degree of the line bundle is $2k - 2n = 0$ and thus $n = k$. Hence a conic in $\mathbf{C}\mathrm{P}^2$ lifts to $P(V_k)$ if and only if it has the property that

$$\mathcal{O}(k/2, -k/2) \cong \mathcal{O} \quad \text{on} \quad \tilde{C}.$$

Now recall the Poncelet problem [3]: to find a polygon with k sides which is inscribed in a conic C and circumscribed about a conic B. The projection

$$\pi : \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1 \to \mathbf{C}\mathrm{P}^2$$

we have already used is the correct setting for the problem.

Let (a,b) be a point in $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$ and consider the two lines $\{a\} \times \mathbf{C}\mathrm{P}^1$ and $\mathbf{C}\mathrm{P}^1 \times \{b\}$ passing through it. The first line is a divisor of the linear system $\mathcal{O}(1,0)$ and the second of $\mathcal{O}(0,1)$. As we have seen, their images in $\mathbf{C}\mathrm{P}^2$ are the two tangents to the branch conic B from the point $\pi(a,b)$. Now let C be the conic which contains the vertices of the Poncelet polygon, and let $P_1 = (a_1, b_1) \in \tilde{C} \subset \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$ be a point lying over an initial vertex. The line $\{a_1\} \times \mathbf{C}\mathrm{P}^1$ meets $\tilde{C} \sim \mathcal{O}(2,2)$ in two points generically, which are P_1 and a second point $P_2 = (a_1, b_2)$. The two points $\pi(P_1)$ and $\pi(P_2)$ lie on C, and the line joining them is $\pi(\{a_1\} \times \mathbf{C}\mathrm{P}^1)$ which is tangent to B, and hence is a side of the polygon. The other side of the polygon through $\pi(P_2)$ is $\pi(\mathbf{C}\mathrm{P}^1 \times \{b_2\})$ which meets the conic C at $\pi(P_3) = \pi(a_2, b_2)$. We carry on this procedure using the two lines through each point, to obtain $P_1, \ldots, P_{k+1}$. Since the Poncelet polygon is closed with k vertices, we have $\pi(P_{k+1}) = \pi(P_1)$.

Consider now the divisor classes $P_i + P_{i+1}$. We have

$$\begin{array}{rcl} P_1 + P_2 & \sim & \mathcal{O}(1,0) \\ P_2 + P_3 & \sim & \mathcal{O}(0,1) \\ P_3 + P_4 & \sim & \mathcal{O}(1,0) \\ \ldots & & \ldots \end{array}$$

and $P_k + P_{k+1} \sim \mathcal{O}(1,0)$ if k is odd and $\sim \mathcal{O}(0,1)$ if k is even.

In the odd situation, taking the alternating sum we obtain

$$P_1 + P_{k+1} \sim \mathcal{O}((k+1)/2, -(k-1)/2) \tag{10}$$

and since $\pi(P_{k+1}) = \pi(P_1)$, then $P_{k+1} = P_1$ or $\sigma(P_1)$. However, in the former case, we would have

$$P_k + P_1 \sim P_k + P_{k+1} \sim \mathcal{O}(1,0) \sim P_1 + P_2$$

and consequently $P_2 \sim P_k$ on the elliptic curve $\tilde{C}$ which implies $P_2 = P_k$. But $\pi(P_k)$ and $\pi(P_2)$ are different vertices of the polygon, so we must have $P_{k+1} = \sigma P_1$. This implies that the divisor $P_{k+1} + P_1 = \pi^{-1}(\pi(P_1))$ and so in the notation above

$$P_{k+1} + P_1 \sim H^{1/2} = \mathcal{O}(1/2, 1/2).$$

From (10) we therefore obtain the constraint on $\tilde{C}$

$$\mathcal{O}(k/2, -k/2) \sim \mathcal{O} \tag{11}$$

which is exactly the condition for the conic to lift to $P(V_k)$. A similar argument leads to the same condition for k even, where in this case $P_{k+1} = P_1$.

In the case that $d = 1$, C is a line, but the argument is very similar. Here $M \cong H^n$ for some n and on $\tilde{C}$ we have a section ξ of $\mathcal{O}(k-n, -n)$. This time, since $P.D_2 = 2$, the line bundle is of degree 2, so $k - 2n = 2$, and so a lifting is defined by a section of $\mathcal{O}(1 + k/2, 1 - k/2)$ on $\tilde{C}$.

Example

Let us now compare this interpretation with the equivariant compactification $\mathbf{C}\mathrm{P}^3$ of $SL(2, \mathbf{C})/\tilde{D}_3$ discussed earlier. In the first place, consider the line bundle

$$\tilde{U} = U - 2H$$

on $P(V_3)$. Now since for any 2-dimensional vector space $V^* \cong V \otimes \Lambda^2 V^*$, $P(V_k) = P(V_k^*)$, but with different tautological bundles. The tautological bundle for $P(V_3^*)$ is actually $\tilde{U}$, and so there are canonical isomorphisms

$$\begin{aligned} H^0(P(V_3), -\tilde{U}) &\cong H^0(\mathbf{C}\mathrm{P}^2, V_3) \cong H^0(\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1, \mathcal{O}(3,0)) \\ &\cong H^0(\mathbf{C}\mathrm{P}^1, \mathcal{O}(3)) \cong \mathbf{C}^4. \end{aligned}$$

The linear system $|-\tilde{U}|$ therefore maps $P(V_3)$ equivariantly to $\mathbf{C}\mathrm{P}^3$. Since $P.D_1 = 4$ and $P.D_2 = 0$, it follows from (8) and (9), that $P.H = 2$ and $P.U = 3$, and so $P.\tilde{U} = -1$, so under this mapping the curves P map to projective lines.

There is a more geometric way of seeing the relation of lines in $\mathbf{C}\mathrm{P}^3$ to Poncelet triangles. Recall that we are viewing $\mathbf{C}\mathrm{P}^3$ as the space of cubic polynomials, and $\mathbf{C}\mathrm{P}^2$ as the space of quadratic polynomials. The quadratics with a fixed linear factor $z - \alpha$ describe, as we have seen, a line in $\mathbf{C}\mathrm{P}^2$ which is tangent to the discriminant conic at the quadratic $(z-\alpha)^2$. Thus the three linear factors of a cubic $(z-\alpha), (z-\beta), (z-\gamma)$ describe the sides of a triangle circumscribing the conic B. Its vertices are the quadratic factors $(z-\alpha)(z-\beta), (z-\beta)(z-\gamma), (z-\gamma)(z-\alpha)$.

Now consider a straight line of cubics $p_t(z) = tp(z) + q(z)$ with roots $\alpha(t), \beta(t)$ and $\gamma(t)$. We have a 1-parameter family of triangles and

$$\begin{aligned} tp(\alpha) + q(\beta) &= 0 \\ tp(\beta) + q(\beta) &= 0. \end{aligned}$$

Now from these two equations

$$0 = p(\alpha)q(\beta) - p(\beta)q(\alpha) = (\alpha - \beta) r(\alpha, \beta)$$

where $r(\alpha, \beta)$ is a symmetric polynomial in α, β. It is in fact *quadratic* in $\alpha + \beta, \alpha\beta$ and thus defines a conic C in the plane.

Hence, as t varies, the vertices of the triangle lie on a fixed conic C, and we have a solution of the Poncelet problem for $k = 3$.

5 Solutions of Painlevé VI

To find more about the connection we have just defined on $Z = P(V_k)$ entails descending to local coordinates, which we do next.

Consider the projective bundle $P(V_k)$ pulled back to $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$. At a point off the diagonal $(a, b) \in \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$, as in (7), the fibre is

$$P(\mathcal{O}(k,0)_a \oplus \mathcal{O}(k,0)_b) = P(\mathcal{O}(k,0) \oplus \mathcal{O}(0,k))_{(a,b)}$$

and away from the zero section of the second factor, this is isomorphic to

$$\mathcal{O}(k,-k)_{(a,b)}.$$

Now choose standard affine coordinates (w, z) in $\mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$. Since $K_{\mathbf{C}\mathrm{P}^1} \cong \mathcal{O}(-2)$, we have corresponding local trivializations dw and dz of $\mathcal{O}(-2,0)$ and $\mathcal{O}(0,-2)$. These define a local trivialization $(dw)^{-k/2}(dz)^{k/2}$ of $\mathcal{O}(k,-k)$, and thus coordinates

$$(w, z, s) \mapsto s(dw)^{-k/2}(dz)^{k/2}_{(w,z)}.$$

Note that Z is the quotient of this space by the involution $(w, z, s) \mapsto (z, w, s^{-1})$. From this trivialization, the natural action of $SL(2, \mathbf{C})$ on differentials gives the action on Z:

$$(w, z, s) \mapsto \left(\frac{aw+b}{cw+d}\, , \quad \frac{az+b}{cz+d}\, , \quad \frac{(cz+d)^k}{(cw+d)^k} s \right).$$

Differentiating this expression at the identity gives the tangent vector (w', z', s') corresponding to a matrix

$$\begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} \in \underline{g}$$

as

$$\begin{aligned} w' &= -c'w^2 + 2a'w + b' \\ z' &= -c'z^2 + 2a'z + b' \\ s' &= -kc'(w-z)s \end{aligned}$$

This is $\alpha(a', b', c') \in TZ_{(w,z,s)}$. Solving for (a', b', c') gives the entries of the matrix of 1-forms $A = \alpha^{-1}$ as

$$\begin{aligned} A_{11} &= \frac{dw - dz}{2(w-z)} - \frac{(w+z)ds}{2ks(w-z)} \\ A_{12} &= \frac{wdz - zdw}{(w-z)} + \frac{wzds}{ks(w-z)} \\ A_{21} &= -\frac{ds}{ks(w-z)} \end{aligned} \tag{12}$$

Proposition 4 *The residue of the connection at a singular point is conjugate to*

$$\begin{pmatrix} 1/4 & 0 \\ 0 & -1/4 \end{pmatrix} \quad \textit{on } D_1 \qquad \textit{and} \qquad \begin{pmatrix} 1/2k & 0 \\ 0 & -1/2k \end{pmatrix} \quad \textit{on } D_2$$

Proof: In these coordinates, $s = 0$ is the equation of D_2. From (2), the residue of A at $s = 0$ is

$$\begin{pmatrix} -(w+z)/2k(w-z) & wz/k(w-z) \\ -1/k(w-z) & (w+z)/2k(w-z) \end{pmatrix} \tag{13}$$

which has determinant $-1/4k^2$ and therefore eigenvalues $\pm 1/2k$.

To find the residue at D_1, we need different coordinates, since the above ones are invalid on the diagonal. Take the affine coordinates $x = -(w + z), y = wz$ on $\mathbf{C}\mathrm{P}^2$. Since the holomorphic functions in w, z form a module over the symmetric functions generated by $1, w - z$ we can use these to give coordinates in the projectivized direct image $P(V_k)$, which are valid for $w = z$. We obtain an affine fibre coordinate t related to s above by

$$s = \frac{t + w - z}{t - w + z}.$$

Using this and local coordinates x and $u = (w - z)^2 = x^2 - 4y$ on $\mathbf{C}\mathrm{P}^2$ the divisor D_1 is given by $u = 0$ and the residue there is

$$\begin{pmatrix} 1/4 + x/2kt & x/4 + x^2/4kt \\ -1/kt & -1/4 - x/2kt \end{pmatrix} \tag{14}$$

This has determinant $-1/16$ and hence eigenvalues $\pm 1/4$.

Remark

Exponentiating the residues we see that the holonomy of a small loop around the divisor D_1 or D_2 is conjugate to:

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \text{on } D_1 \qquad \text{and} \qquad \begin{pmatrix} e^{i\pi/k} & 0 \\ 0 & e^{-i\pi/k} \end{pmatrix} \quad \text{on } D_2$$

In the dihedral group $D_k \subset SO(3)$ the conjugacy classes are those of a reflection in the plane and a rotation by $2\pi/k$.

These facts tell us something of the structure of the divisor D_1. Since we know that the residue of the meromorphic connection at a singular point lies in the Lie algebra of the stabilizer of the point, and this is here semisimple, the orbit is isomorphic to

$$SL(2, \mathbf{C})/\mathbf{C}^* \cong \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1 \backslash \Delta$$

where Δ is the diagonal. The projections onto the two factors must, by $SL(2, \mathbf{C})$ invariance, be the two eigenspaces of the residue corresponding to the eigenvalues $\pm 1/4$.

Now $D_1 = \pi^{-1}(B)$ is a projective bundle over the conic $B \cong \mathbf{C}\mathrm{P}^1$. By invariance it must be one of the factors above. To see which, note that from (14), eigenvectors for the eigenvalues $1/4$ and $-1/4$ are respectively

$$\begin{pmatrix} x+kt \\ -2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x \\ -2 \end{pmatrix}$$

and so, from the choice of coordinates above, clearly the second represents the projection to B. Note, moreover, that on the diagonal $w = z$, the coordinate $x = -(w+z) = -2z$, so that x is an affine parameter on $B \cong \Delta \cong \mathbf{C}\mathrm{P}^1$. Furthermore, when $x = \infty$ the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of the residue with eigenvalue $-1/4$.

Now let us use this information to determine the solution to the Painlevé equation corresponding to a rational curve $P \subset Z$. Recall that the curve $C = \pi(P)$ is a plane curve of degree d, where $d = 1$ or $d = 2$. As we have seen, when $d = 1$, any line is of this form, but when $d = 2$, the conic must circumscribe a Poncelet polygon.

By the $SL(2, \mathbf{C})$ action, we can assume that C meets the conic B at the point $x = \infty$. From the discussion above, if A_∞ is the residue of the connection at this point, then

$$A_\infty \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

From Proposition 1, the solution of the Painlevé equation is the point y on the curve P at which $A(y)$ has this same eigenvector, i.e. where

$$A_{21}(y) = 0$$

.

Proposition 5 *A line in the plane defines a solution to Painlevé's sixth equation with coefficients* $(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/2k^2, 1/2 - 1/2k^2)$. *A Poncelet conic in the plane defines a solution to the Painlevé equation with coefficients* $(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/8, 3/8)$.

Proof: The residues on D_1 and D_2 are given by (14) and (13). The lifting of a line meets D_1 and D_2 in two points each, so using (6) (and taking account of the fact that the roles of the two basis vectors are interchanged), we obtain the first set of coefficients. The lifting of a Poncelet conic meets D_1 in four points, which gives the second set, again from (6).

Proposition 6 *The lifting of a line in* $\mathbf{C}\mathrm{P}^2$ *to* $P(V_k)$ *defines the solution*

$$y = \sqrt{x}$$

of Painlevé VI with $(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/2k^2, 1/2 - 1/2k^2)$.

Proof: Taking the double covering $\tilde{C} \subset \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$ of the line C and the coordinates w, z, s on the corresponding covering of Z, the lifted curve P is defined locally by a function s on the curve $\tilde{C}$. In fact, as we shall see next, s is a meromorphic function on $\tilde{C}$ with certain properties.

From the comments following Theorem 2, the lifting is given by a holomorphic section ξ of $\mathcal{O}(1+k/2, 1-k/2)$. This line bundle has degree 2 on $\tilde{C}$, and so ξ vanishes at two points. Applying the involution σ, then $\sigma^*\xi$ is a section of $\mathcal{O}(1-k/2, 1+k/2)$. Considering ξ as a section of $\mathcal{O}(k, 0) \otimes \mathcal{O}(1-k/2, 1-k/2)$, the lifting of C to $P(V_k)$ is defined by $(\xi_{(a,b)}, \xi_{(b,a)})$, or in the coordinates w, z, s,

$$s(dw)^{-k/2}(dz)^{k/2} = \xi/\sigma^*\xi. \tag{15}$$

Since $dw^{-1/2}$ and $dz^{-1/2}$ are holomorphic sections of $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$, it follows that on $\tilde{C}$, s is a meromorphic function. Now using the $SL(2, \mathbf{C})$ action, we may assume that the line C is given by $x = 0$, which means that $\tilde{C}$ has equation

$$w = -z$$

which defines an obvious trivialization of $\mathcal{O}(k, -k)$ and from which we deduce that s has two simple zeros at $(a_1, -a_1), (a_2, -a_2)$ and two poles at $(-a_1, a_1), (-a_2, a_2)$. Using z as an affine parameter on $\tilde{C}$, we obtain, up to a constant multiple,

$$s = \frac{(z + a_1)(z + a_2)}{(z - a_1)(z - a_2)}. \tag{16}$$

Now $y = wz$ is an affine parameter on the line C, which meets the conic B at $y = 0, \infty$. The lifting P meets the divisor D_2 where $y = -a_1^2$ and $y = -a_2^2$, so putting $a_1 = i$ and $a_2 = \sqrt{-x}$, then P is a projective line with a parametrization such that the singular points of the induced connection are $0, 1, x, \infty$, as required for the Painlevé equation.

It remains to determine the solution of the equation, which is given by $A_{21}(y) = 0$. But from (12), this is where $ds = 0$, and from (16) this is equivalent to

$$\frac{1}{z + a_1} - \frac{1}{z - a_1} + \frac{1}{z + a_2} - \frac{1}{z - a_2} = 0$$

which gives

$$y = -z^2 = -a_1 a_2 = \sqrt{x}$$

with the above choices of a_1, a_2.

Remarks

1. By direct calculation, the function $y = \sqrt{x}$ solves Painlevé VI for any coefficients satisfying $\alpha + \beta = 0$ and $\gamma + \delta = 1/2$. From (6) this occurs when the residues are conjugate in pairs.

2. When $k = 2$, we obtain $(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/8, 3/8)$ which are the coefficients arising from Poncelet conics. We shall see the same solution appearing in the next section in the context of Poncelet quadrilaterals.

Naturally, the solutions corresponding to Poncelet conics are more complicated, and we shall give some explicitly in Section 6. Here we give the general algebraic procedure for obtaining them.

In the case of a conic C in $\mathbf{C}\mathrm{P}^2$, we have a section ξ, in fact a trivialization, of the bundle $\mathcal{O}(k/2, -k/2)$ on the elliptic curve $\tilde{C}$. As in (15), we still define the lifting by

$$s(dw)^{-k/2}(dz)^{k/2} = \xi/\sigma^*\xi$$

but in this case ξ is non-vanishing. The section $(dw)^{-k/2}$ vanishes to order k at $\infty \in \mathbf{C}\mathrm{P}^1$, and so at the two points $(\infty, \infty), (\infty, b)$ where $\tilde{C}$ meets $\{\infty\} \times \mathbf{C}\mathrm{P}^1$. Similarly $(dz)^{-k/2}$ vanishes at $(\infty, \infty), (b, \infty)$. The meromorphic function s can be regarded as a map of curves

$$s : \tilde{C} \to \mathbf{C}\mathrm{P}^1.$$

It follows that s is a meromorphic function on $\tilde{C}$ with a zero of order k at (b, ∞), a pole of order k at (∞, b) and no other zeros or poles.

The derivative ds is invariantly defined as a section of $K_{\tilde{C}} \otimes s^* K^{-1}_{\mathbf{C}\mathrm{P}^1} \cong s^*\mathcal{O}(2)$ (since $\tilde{C}$ is an elliptic curve and hence has trivial canonical bundle). In particular ds vanishes with total multiplicity $2k$. But since (∞, a) and (a, ∞) are branch points of order k, ds has a zero of order $k-1$ at each of these points, leaving two extra points as the remaining zeros. Since the involution σ takes s to s^{-1}, these points are paired by the involution, and give a single point $y \in \mathbf{C}\mathrm{P}^1$ which is our solution to the Painlevé equation.

Fortunately Cayley's solution in 1853 to the Poncelet problem gives us the means to find y algebraically. A useful modern account of this is given by Griffiths and Harris in [5], but the following description I owe to M. F. Atiyah.

Suppose the elliptic curve $\tilde{C}$ is described as a cubic in $\mathbf{C}\mathrm{P}^2$ given by $v^2 = h(u)$, where $h(u)$ is a cubic polynomial and $h(0) = c_0^2 \neq 0$. We shall find the condition on the coefficients of h in order that there should exist a polynomial $g(v, u)$ of degree $(n-1)$

(a section of $\mathcal{O}(n-1)$) on the curve with a zero of order $(2n-1)$ at $(v,u)=(c_0,0)$ and a pole of order $(n-2)$ at $u=\infty$. Given such a polynomial,

$$s(v,u)=\frac{g(v,u)}{g(-v,u)}$$

is the function on the curve (for $k=(2n-1)$) used above, and the zeros of its derivative define the solution $y(x)$ to the Painlevé equation. A very similar procedure deals with the case of even k.

To find g, expand $\sqrt{h(u)}$ as a power series in z, making a choice c_0 of square root of $h(0)$:

$$v=c_0+c_1u+c_2u^2+\ldots$$

and then put

$$v_m=c_0+c_1u+\ldots+c_{m-1}u^{m-1}.$$

Now clearly $v-v_n$ has a zero of order n on the curve at $u=0$, as do other functions constructed from the v_m:

$$\begin{aligned}
v-v_n &= c_nu^n+\ldots+c_{2n-2}u^{2n-2}+\ldots\\
u(v-v_{n-1}) &= c_{n-1}u^n+\ldots+c_{2n-3}u^{2n-2}+\ldots\\
u^2(v-v_{n-2}) &= c_{n-2}u^n+\ldots+\ldots\\
\ldots &= \ldots\\
u^{n-2}(v-v_2) &= c_2u^n+\ldots+c_nu^{2n-2}+\ldots
\end{aligned}$$

We can then find $n-1$ coefficients $\lambda_0,\lambda_1,\ldots,\lambda_{n-2}$ such that

$$g(v,u)\equiv\lambda_0(v-v_n)+\lambda_1u(v-v_{n-1})+\ldots+\lambda_{n-2}u^{n-2}(v-v_2)$$

vanishes at $u=0$ to order $2n-1$ if and only if

$$\det M=0\quad\text{where}\quad M=\begin{bmatrix} c_n & c_{n-1} & \ldots & c_2\\ c_{n+1} & c_n & \ldots & c_3\\ \ldots & \ldots & \ldots & \ldots\\ c_{2n-2} & c_{2n-3} & \ldots & c_n\end{bmatrix}\tag{17}$$

This is Cayley's form of the Poncelet constraint.

If (17) holds, $g(v,u)$ is a polynomial of degree $n-1$ which, upon inspection, vanishes with multiplicity $n-2$ at the inflexion point at infinity of the curve. Its total intersection number with the cubic $\tilde{C}$ is $3(n-1)=(2n-1)+(n-2)$, so there are no more zeros. Thus the condition $\det M=0$ is necessary and sufficient for the construction of the required function s with a zero of order k at $(v,u)=(c_0,0)$ and a pole of order k at $(v,u)=(-c_0,0)$.

In the case of a pair of conics in the plane, defined by symmetric matrices B and C, the constraint is on the cubic $h(u)=\det(B+uC)$ in order for the conics to satisfy the Poncelet condition.

Note that

$$g(v,u)=p(u)v+q(u)$$

where p and q are polynomials of degree $n-2$ and $n-1$ respectively. Thus

$$s=\frac{p(u)v+q(u)}{-p(u)v+q(u)}$$

and ds vanishes if

$$pqv'+(p'q-pq')v=0.$$

Using $v^2=h(u)$, this is equivalent to

$$r(u)\equiv p(u)q(u)h'(u)+2(p'(u)q(u)-p(u)q'(u))h(u)=0$$

This is a polynomial in u of degree $2n-1$, which by construction vanishes to order $k-1=2n-2$ at $u=0$. It is thus of the form

$$r(u)=au^{2n-2}(u-b)$$

and so y, the solution to the Painlevé equation which corresponds to a zero of ds, is defined in terms of the ratio of the two highest coefficients of $r(u)$. Since the solution to the Painlevé equation has singularities at the four points $0,1,x,\infty$, a Möbius transformation gives the variable x, and the solution $y(x)$, as:

$$x=\frac{e_3-e_1}{e_2-e_1}\qquad y=\frac{b-e_1}{e_2-e_1} \tag{18}$$

where e_1,e_2 and e_3 are the roots of $h(u)=0$.

To calculate b explicitly is easy. Putting $p(u)=p_0+p_1u+\ldots+p_{n-2}u^{n-2}$ and $q(u)=q_0+q_1u+\ldots q_{n-1}u^{n-1}$ and looking at the coefficients of $r(u)$, we find

$$\begin{aligned} b &= \frac{p_{n-3}}{p_{n-2}}-3\frac{q_{n-2}}{q_{n-1}} \\ &= \frac{\lambda_{n-3}}{\lambda_{n-2}}-3\frac{\lambda_0c_{n-2}+\lambda_1c_{n-3}+\ldots+\lambda_{n-2}c_0}{\lambda_0c_{n-1}+\lambda_1c_{n-2}+\ldots+\lambda_{n-2}c_1} \end{aligned}$$

from the definition of $p(u)$ and $q(u)$. Now the coefficients λ_i are the entries of a column vector λ such that $M\lambda = 0$. Thus in the generic case where the rank of M is $n-3$, these are given by cofactors of M . We can then write

$$b = -\frac{\begin{vmatrix} c_n & \dots & c_4 & c_2 \\ c_{n+1} & \dots & c_5 & c_3 \\ \dots & \dots & \dots & \dots \\ c_{2n-3} & \dots & c_{n+1} & c_{n-1} \end{vmatrix}}{\begin{vmatrix} c_n & c_{n-1} & \dots & c_3 \\ c_{n+1} & c_n & \dots & c_4 \\ \dots & \dots & \dots & \dots \\ c_{2n-3} & c_{2n-4} & \dots & c_n \end{vmatrix}} - 3\frac{\begin{vmatrix} c_n & c_{n-1} & \dots & c_2 \\ \dots & \dots & \dots & \dots \\ c_{2n-3} & c_{2n-4} & \dots & c_{n-1} \\ c_{n-2} & c_{n-3} & \dots & c_0 \end{vmatrix}}{\begin{vmatrix} c_n & c_{n-1} & \dots & c_2 \\ \dots & \dots & \dots & \dots \\ c_{2n-3} & c_{2n-4} & \dots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 \end{vmatrix}} \qquad (19)$$

This effectively gives us a concrete form for the solution of the Painlevé equation for k odd, though we shall try to be more explicit in special cases in the next section. When k is even, a similar analysis can be applied. Very briefly, if $k = 2n$, and $n \geq 2$, then the vanishing of

$$\begin{vmatrix} c_{n+1} & c_n & \dots & c_3 \\ c_{n+2} & c_{n+1} & \dots & c_4 \\ \dots & \dots & \dots & \dots \\ c_{2n-1} & \dots & \dots & c_{n+1} \end{vmatrix}$$

is the condition for the existence of λ_i so that

$$g(v,u) = \lambda_0(v - v_{n+1}) + \lambda_1 u(v - v_n) + \ldots + \lambda_{n-2}u^3(v - v_3)$$

has a zero of order $2n$ at $u = 0$. The rest follows in a similar manner to the above.

6 Explicit solutions

We shall now calculate explicit solutions to Painlevé VI with coefficients $(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/8, 3/8)$ for small values of k. Clearly, from the interpretation in terms of Poncelet polygons, we must have $k \geq 3$. The discussion in the previous section shows that we need to perform calculations with the coefficients of the cubic $h(u) = \det(B + uC)$ where B and C are symmetric matrices representing the conics we have denoted with the same symbol. For convenience, we take the cubic

$$\begin{aligned} h(u) &= (1 + (x_1 + x_2)u)(1 + (x_2 + x_0)u)(1 + (x_0 + x_1)u) \\ &= 1 + 2s_1 u + (s_1^2 + s_2)u^2 + (s_1 s_2 - s_3)u^3 \qquad (20) \end{aligned}$$

where s_i is the ith elementary symmetric function in x_0, x_1, x_2.

6.1 Solution for $k = 3$

For $k = 3$ the Poncelet constraint from (17) is $c_2 = 0$, which is $s_2 = 0$ for the above cubic, and can therefore be written

$$\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} = 0.$$

This is the equation in homogeneous coordinates (x_0, x_1, x_2) for a plane conic which can clearly be parametrized rationally by

$$x_0 = \frac{1}{1+s} \qquad x_1 = \frac{-1}{s} \qquad x_2 = -1 \tag{21}$$

The polynomial g is just $g(v, u) = v - (1 + s_1 u)$, and this gives

$$r(u) = s_1 s_3 u^3 + 3 s_3 u^2$$

and hence $b = -3/s_1$. Substituting the parametrization (21) in (18) and using the fact that the roots of $h(u) = 0$ are $u = -1/(x_1 + x_2)$ etc, gives the solution $y(x)$ to the Painlevé equation as

$$y = \frac{s^2(2s^2 + 5s + 2)}{(2s+1)(s^2 + s + 1)} \quad \text{where} \quad x = \frac{s^3(s+2)}{2s+1}$$

6.2 Solution for $k = 4$

The Poncelet constraint here is $c_3 = 0$, which in the formalism above is $s_3 = 0$, that is

$$x_0 = 0 \qquad x_1 = 0 \qquad x_2 = 0.$$

In $\mathbf{C}\mathrm{P}^2$ this consists of three lines. Take the component $x_0 = 0$ and parametrize it by

$$x_1 = 1 \qquad x_2 = s.$$

Now the polynomial g is given by $g(v, u) = v - (1 + s_1 u + \frac{1}{2} s_2 u^2)$, which yields

$$r(u) = \frac{1}{2} s_1 s_2^2 u^4 + s_2^2 u^3$$

and hence $b = -2/s_1$. Substituting the parametrization, we obtain $x = s^2$, $y = s$, thus the solution to the Painlevé equation is

$$y(x) = \sqrt{x}.$$

Remark

In Proposition 6, we saw that the same solution $y = \sqrt{x}$ with coefficients $(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/8, 3/8)$ arises from taking a curve P in $P(V_2)$ with $P.D_1 = P.D_2 = 2$, and hence has holonomy in $\tilde{D}_2$, which is the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, a proper subgroup of $\tilde{D}_4$. Recall also that from [16], $P(V_2) \cong P(T)$, the projectivized tangent bundle of $\mathbf{C}\mathrm{P}^2$. In geometrical terms, a point of $P(T)$ consists of a point in $\mathbf{C}\mathrm{P}^2$ together with a line passing through it, or equivalently a line with a distinguished point. Thus there is a projection also to the dual projective plane $\mathbf{C}\mathrm{P}^{2*}$. In other words

$$P(T) = \{(p, L) \in \mathbf{C}\mathrm{P}^2 \times \mathbf{C}\mathrm{P}^{2*} : p \in L\}$$

with projections onto the two factors. In the terminology above, the two corresponding hyperplane divisor classes are H and $H - U$. Now the curve P which defines the solution to the Painlevé equation satisfies $P.H = P.(H - U) = 1$. It follows easily that P is obtained by taking a point $q \in \mathbf{C}\mathrm{P}^2$ and a skew line M. The set

$$P = \{(p, L) \in P(T) : q \in L \quad \text{and} \quad p = L \cap M\}$$

describes the rational curve $P \subset P(T)$.

According to our formula, this curve must correspond to a Poncelet conic. In fact, let (p, L) be a point of P, and let ℓ denote the pole of the line L with respect to the conic B. The line ℓp has pole q, and as L varies in the pencil of lines through p, the point q describes a conic C. If L is a tangent through p to the conic B, then q is the point of contact $B \cap L$. Let q_1 and q_2 be the two points of contact of the tangents through p, then the conic C passes through p, q_1 and q_2. But pq_1pq_2 is then a degenerate Poncelet quadrilateral, and by Poncelet's theorem [3], if there is a Poncelet polygon through one point of C, then there exists one through each point.

6.3 Solution for $k = 5$

From (17), the Poncelet constraint is

$$\det \begin{bmatrix} c_3 & c_2 \\ c_4 & c_3 \end{bmatrix} = 0$$

which in terms of the symmetric functions s_i is

$$4s_3^2 + s_2^3 - 4s_1s_2s_3 = 0. \tag{22}$$

Now from (19),

$$\begin{aligned} b &= -\frac{c_2}{c_3} - 3\frac{(c_3 - c_1c_2)}{(c_1c_3 - c_2^2)} \\ &= \frac{s_2}{s_3} - 6\frac{(s_3 + s_1s_2)}{(2s_1s_3 + s_2^2)} \end{aligned}$$

and using (22) this becomes

$$b = -20\frac{s_2s_3}{(4s_3^2 + 3s_2^3)}.$$

It is convenient to introduce coordinates u, v by setting

$$x_0 = 1 \qquad u = \frac{1}{x_1} + \frac{1}{x_2} \qquad v = \frac{1}{x_1x_2}$$

and then the constraint (22) becomes

$$v = \frac{(1+u)(1-u)^2}{4u} \tag{23}$$

and

$$b = -5\frac{(u+1)(u-1)^2}{(3u(u+1)^2 + (u-1)^2)}$$

Now $1/x_1$ and $1/x_2$ are the roots of the quadratic $z^2 - uz + v = 0$, and so, using (23),

$$\frac{1}{x_1}, \frac{1}{x_2} = \frac{1}{2}(u \pm \sqrt{1 + u - u^{-1}})$$

and now putting

$$w^2 = 1 + u - u^{-1} \tag{24}$$

we finally obtain the solution of the Painlevé equation as

$$\begin{aligned} y &= \frac{(u-w+2)(u+w)}{(u+w-2)(u-w)}\left(1 - \frac{20u^2}{(3u(u+1)^2 + (u-1)^2)}\right) \\ \text{where} \quad x &= \frac{(u-w-2)(u-w+2)(u+w)^2}{(u+w-2)(u+w+2)(u-w)^2} \quad \text{and} \quad w^2 = 1 + u + u^{-1} \end{aligned}$$

Note that (24) is the equation of an elliptic curve, so that x and y are meromorphic functions on the curve. It is a special elliptic curve, in fact under the Cremona transformation $x_i \mapsto 1/x_i$, the equation (22) transforms into the plane cubic

$$s_1^3 - 4s_1s_2 + 4s_3 = 0$$

and the symmetric group S_3 clearly acts as automorphisms of the curve.

The study of the Poncelet constraints for small values of k is undertaken in [1], and the reader will find that, apart from $k = 6$ and $k = 8$, the formulae rapidly become more complicated. We shall only consider now these two further cases.

6.4 Solution for $k = 6$

Here, from [1], we find that the constraint factorizes

$$(x_0x_1+x_1x_2+x_2x_0)(-x_0x_1+x_1x_2+x_2x_0)(x_0x_1-x_1x_2+x_2x_0)(x_0x_1+x_1x_2-x_2x_0) = 0.$$

The first factor represents the case $k = 3$ embedded in $k = 6$, by thinking of a repeated Poncelet triangle as a hexagon. We choose instead the third factor, which can be written as

$$\frac{1}{x_0} = \frac{1}{x_1} + \frac{1}{x_2}.$$

This is a conic, and we rationally parametrize it by setting

$$x_0 = \frac{1}{1+s} \qquad x_1 = \frac{1}{s} \qquad x_2 = 1.$$

After some calculation, this gives a solution to the Painlevé equations

$$\begin{aligned} y &= \frac{s(1+s+s^2)}{(2s+1)} \\ \text{where} \quad x &= \frac{s^3(s+2)}{(2s+1)} \end{aligned}$$

6.5 Solution for $k = 8$

Here, again referring to [1], we find the constraint equation splits into components

$$(-x_0^2x_1^2 + x_1^2x_2^2 + x_2^2x_0^2)(x_0^2x_1^2 - x_1^2x_2^2 + x_2^2x_0^2)(x_0^2x_1^2 + x_1^2x_2^2 - x_2^2x_0^2) = 0.$$

one of which is given by the equation

$$\frac{1}{x_0^2} = \frac{1}{x_1^2} + \frac{1}{x_2^2}$$

and, parametrizing this conic rationally in the usual way with

$$x_0 = 1 \qquad x_1 = \frac{1+s^2}{2s} \qquad x_2 = \frac{1+s^2}{1-s^2}$$

one may obtain the solution to the Painlevé equation as

$$\begin{aligned} y &= \frac{4s(3s^2-2s+1)}{(1+s)(1-s)^3(s^2+2s+3)} \\ \text{where} \quad x &= \left(\frac{2s}{1-s^2}\right)^4. \end{aligned}$$

7 Painlevé curves

The examples above show that there do indeed exist algebraic solutions to Painlevé's sixth equation, for certain values of the coefficients, despite the general description of solutions to these equations as "Painlevé transcendants". In general, an algebraic solution is given by a polynomial equation

$$R(x, y) = 0$$

which defines an algebraic curve. So far, we have only seen explicit examples where this curve is rational or elliptic, but higher genus curves certainly do occur. We make the following definition:

Definition 2 *A Painlevé curve is the normalization of an algebraic curve* $R(x, y) = 0$ *which solves Painlevé's sixth equation (5) for some values of the coefficients* $(\alpha, \beta, \gamma, \delta)$*.*

Just as the elliptic curve above corresponding to the solution for $k = 5$ was special, so are Painlevé curves in general. The equation (5) was in fact found not by Painlevé, but by R. Fuchs [4], but nevertheless falls into the Painlevé classification by its characteristic property that its solutions have no "movable singular points". What this means is that the branch points or essential singularities of solutions $y(x)$ are independent of the constants of integration. In the case of Painlevé VI, these points occur only at $x = 0, 1, \infty$. Now if X is a Painlevé curve, x and y are meromorphic functions on X, and so there are no essential singularities. The function

$$x : X \to \mathbf{C}\mathrm{P}^1$$

is thus a map with branch points only at $x = 0, 1$ and ∞.

Such curves have remarkable properties. In the first place, it follows from Weil's rigidity theorem [17], that X is defined over the algebraic closure $\bar{\mathbf{Q}}$ of the rationals (from Belyi's theorem [2] this actually characterizes curves with such functions). Secondly, by uniformizing $\mathbf{C}\mathrm{P}^1\backslash\{0,1,\infty\}$,

$$X \cong \overline{H/\Gamma}$$

where H is the upper half-plane and Γ is a subgroup of finite index in the principal congruence subgroup $\Gamma(2) \subset SL(2,\mathbf{Z})$. Thus, in some manner, each algebraic solution of Painlevé VI gives rise to a modular curve, and hence corresponds to a problem involving elliptic curves.

Coincidentally, the investigation of the Poncelet problem by Barth and Michel in [1] proceeds by studying a modular curve. This is a curve which occurs in parametrizing elliptic curves with

- a level-2 structure
- a primitive element of order k

In our model of the Poncelet problem, the elliptic curve is $\tilde{C}$, the level-2 structure identifies the elements of order two (or equivalently an ordering of the branch points of π), and the Poncelet constraint (11) selects the line bundle $\mathcal{O}(1/2,-1/2)$ of order k on $\tilde{C}$, or equivalently a point of order k on the curve, the zero of the function s in Section 5. Choosing a primitive element avoids recapturing a solution for smaller k.

As described by Barth and Michel, the stabilizer of a primitive element of order k is

$$\Gamma_{00}(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) : a \equiv d \equiv 1(k) \quad \text{and} \quad c \equiv 0(k) \right\} \tag{25}$$

Now if k is odd, matrices $A \in \Gamma_{00}(k)$ can be chosen such that A mod 2 is any element of $SL(2,\mathbf{Z_2})$. Thus $\Gamma_{00}(k)$ acts transitively on the level-2 structures. Consider in this case the modular curve

$$X_{00}(k,2) = \overline{H/\Gamma_{00}(k) \cap \Gamma(2)} \tag{26}$$

Since $-I \in \Gamma_{00}(k)$ acts trivially on H, this is a curve parametrizing opposite pairs of primitive elements of order k, and level-2 structures.

When k is even, however, only the two matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

are obtained by reducing mod 2 from $\Gamma_{00}(k)$. Thus the group has three orbits on the level-2 structures, and the curve $X_{00}(k,2)$ has three components each given by (26).

There is another curve in the picture: the curve Π_k defined by the Cayley constraint (17), with the c_i symmetric functions of (x_0, x_1, x_2) as defined by (20). This is a plane curve in homogeneous coordinates (x_0, x_1, x_2). Barth and Michel show that a birational image of $X_{00}(k, 2)$ lies as a union of components of Π_k. In the examples of Section 6, we have already seen this curve, connected for $k = 3, 5$ but with different components for $k = 4, 6, 8$.

In the algebraic construction of y in Section 5, it is clear from (19) that x and y are meromorphic functions on Π_k, and so the Painlevé curve X_k is a rational image of the modular curve. In fact, we have the following

Proposition 7 *The Painlevé curve X_k defined by Poncelet polygons is birationally equivalent to the modular curve $\overline{H/\Gamma_{00}(k) \cap \Gamma(2)}$.*

Proof: Let Y_k be the modular curve, then we already have a map $f : Y_k \to X_k$ as described above. We shall define an inverse on the complement of a finite set.

Let the Painlevé curve be defined by the equation

$$R(x, y) = 0$$

and suppose (x, y) is a point on the curve such that $\partial R/\partial y \neq 0$. Then

$$\frac{dy}{dx} = -\frac{\partial R/\partial x}{\partial R/\partial y}$$

is finite, and thus we can recover the connection matrix $A(z)$ on $\mathbf{C}\mathrm{P}^1 \backslash \{0, 1, x, \infty\}$, its coefficients being rational in $x, y, dy/dx$, by using the formulae for defining the connection from the solution of the Painlevé equation as in [8] (cf Section 2).

Now pull the connection back to the elliptic curve E which is the double covering of $\mathbf{C}\mathrm{P}^1$ branched over the four points. The fundamental group of the punctured elliptic curve consists of the words of even length in the generators $\gamma_1, \gamma_2, \gamma_3$ of $\pi_1(\mathbf{C}\mathrm{P}^1 \backslash \{0, 1, x, \infty\})$. But under the holonomy representation into $SO(3) = SU(2)/\pm 1$ these generators map to reflections in a plane. Thus the even words map to the rotations in the dihedral group, the cyclic group $\mathbf{Z}_k$. Around a singular point in $\mathbf{C}\mathrm{P}^1$, the holonomy is conjugate to

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and so on the double covering branched around the point, the holonomy is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and hence the identity in $SO(3)$.

Thus the holonomy is that of a smooth connection, and so defines an element of $\mathrm{Hom}(\pi_1(E), \mathbf{Z}_k)$. This is a flat line bundle of order k, and through the constructions in Sections 4 and 5 is the same bundle as $\mathcal{O}(1/2, -1/2)$. We have actually made a choice here, since holonomy is determined up to conjugation. There is a rotation in $SO(3)$ which takes the generating rotation of the cyclic group $\mathbf{Z}_k$ to its inverse. Thus x, y defines a pair of $k-$torsion points on the elliptic curve, and hence a single point of the modular curve $Y_k = X_{00}(k, 2)$.

Remark

> A straightforward consideration of the branching over $0, 1, \infty$ leads to the formula
> $$g = \frac{1}{4}(p-3)^2$$
> for the genus g of $X_{00}(p, 2)$ when p is prime (see [1]). Thus the Painlevé curve can have arbitrarily large genus.

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