		Calculated.	Found.
C^{16} . .	96	66·6	65·5
H^{16} . .	16	11·1	11·3
O^4 . .	32	22·3	23·2
		100·0	100·0

Caproate of Magnesia crystallizes in small starry groups of fine needles, and appears like the propionates to contain one equivalent of HO after drying over sulphuric acid. 0·1808 grm. of the dried salt gave, on treating with sulphuric acid, 0·083 grm. of sulphate of magnesia =15·3 per cent. of MgO. The formula

$$MgO . C^{12} H^{11} O^3 + HO$$

requires 15·1 magnesia.

XII. *Note on the Porism of the in-and-circumscribed Polygon.* *By* A. Cayley, *Esq.**

THE equation of a conic passing through the points of intersection of the conics

$$U=0, \quad V=0$$

is of the form

$$wU+V=0,$$

where w is an arbitrary parameter. Suppose that the conic touches a given line, we have for the determination of w a quadratic equation, the roots of which may be considered as parameters for determining the line in question. Let one of the values of w be considered as equal to a given constant k, the line is always a tangent to the conic

$$kU+V=0;$$

and taking $w=p$ for the other value of w, p is a parameter determining the particular tangent, or, what is the same thing, the point of contact of this tangent.

Suppose the tangent meets the conic $U=0$ (which is of course the conic corresponding to $w=\infty$) in the points P, P′, and let θ, ∞ be the parameters of the point P, and θ', ∞ the parameters of the point P′. It follows from my "Note on the Geometrical representation of the Integral $\int dx \div \sqrt{(x+a)(x+b)(x+c)}$ †," and from the theory of invariants, that if $\Box\xi$ represent the "Discriminant" of $\xi U+V$ (I now use the term discriminant in the same sense in which determinant is sometimes used, viz. the

* Communicated by the Author.

† I take the opportunity of correcting an obvious error in the note in question, viz. $a^2+b^2+c^2-2bc-2ca-2ab$ is throughout written instead of (what the expression should be) $b^2c^2+c^2a^2+a^2b^2-2a^2bc-2b^2ca-2c^2ab$.

discriminant of a quadratic function

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

or $(a, b, c, f, g, h)(x, y, z)^2$ is the determinant

$$k=abc-af^2-bg^2-ch^2+2fgh),$$

and if

$$\Pi\xi=\int_\infty \frac{d\xi}{\sqrt{\Box\xi}},$$

then the following theorem is true, viz.

"If (θ, ∞), (θ', ∞) are the parameters of the points P, P′ in which the conic $U=0$ is intersected by the tangent, the parameter of which is p of the conic $kU+V=0$, then the equations

$$\Pi\theta=\Pi p-\Pi k$$

$$\Pi\theta'=\Pi p+\Pi k$$

determine the parameters θ, θ' of the points in question." And again,—

"If the variable parameters θ, θ' are connected by the equation

$$\Pi\theta'-\Pi\theta=2\Pi k,$$

then the line PP′ will be a tangent to the conic $kU+V=0$."

Whence, also,—

"If the sides of a triangle inscribed in the conic $U=0$ touch the conics

$$k\,U+V=0$$

$$k'U+V=0$$

$$k''U+V=0,$$

then the equation

$$\Pi k+\Pi k'+\Pi k''=0$$

must hold good between the parameters k, k', k''."

And, conversely, when this equation holds good, there are an infinite number of triangles inscribed in the conic $U=0$, and the sides of which touch the three conics; and similarly for a polygon of any number of sides.

The algebraical equivalent of the transcendental equation last written down is

$$\begin{vmatrix} 1, & k, & \sqrt{\Box k} \\ 1, & k', & \sqrt{\Box k'} \\ 1, & k'', & \sqrt{\Box k''} \end{vmatrix}=0$$

let it be required to find what this becomes when $k=k'=k''=0$, we have

$$\sqrt{\Box k}=A+Bk+Ck^2+\ldots,$$

and substituting these values, the determinant divides by

$$\begin{vmatrix} 1, & k, & k^2 \\ 1, & k', & k'^2 \\ 1, & k'', & k''^2 \end{vmatrix}$$

the quotient being composed of the constant term C, and terms multiplied by k, k', k''; writing, therefore, $k=k'=k''=0$, we have $C=0$ for the condition that there may be inscribed in the conic $U=0$ an infinity of triangles circumscribed about the conic $V=0$; C is of course the coefficient of ξ^2 in $\sqrt{\square\xi}$, *i. e.* in the square root of the discriminant of $\xi U+V$; and since precisely the same reasoning applies to a polygon of any number of sides,—

Theorem. The condition that there may be inscribed in the conic $U=0$ an infinity of n-gons circumscribed about the conic $V=0$, is that the coefficient of ξ^{n-1} in the development in ascending powers of ξ of the square root of the discriminant of $\xi U+V$ vanishes.

It is perhaps worth noticing that $n=2$, *i. e.* the case where the polygon degenerates into two coincident chords, is a case of exception. This is easily explained.

In particular, the condition that there may be in the conic*

$$ax^2+by^2+cz^2=0$$

an infinity of n-gons circumscribed about the conic

$$x^2+y^2+z^2=0,$$

is that the coefficient of ξ^{n-1} in the development in ascending powers of ξ of

$$\sqrt{(1+a\xi)(1+b\xi)(1+c\xi)}$$

vanishes; or, developing each factor, the coefficient of ξ^{n-1} in

$$\left(1+\frac{1}{2}a\xi-\frac{1}{8}a^2\xi^2+\frac{1}{16}a^3\xi^3-\frac{5}{64}a^4\xi^4+\&c.\right)\left(1+\frac{1}{2}b\xi-\&c.\right)\left(1+\frac{1}{2}c\xi-\&c.\right)$$

vanishes.

Thus, for a triangle this condition is

$$a^2+b^2+c^2-2bc-2ca-2ab=0;$$

for a quadrangle it is

$$a^3+b^3+c^3-bc^2-b^2c-ca^2-c^2a-ab^2-a^2b+2abc=0;$$

which may also be written

$$(b+c-a)(c+a-b)(a+b-c)=0;$$

and similarly for a pentagon, &c.

* I have, in order to present this result in the simplest form, purposely used a notation different from that of the note above referred to, the quantities $ax^2+by^2+cz^2$ and $x^2+y^2+z^2$ being, in fact, interchanged.

Suppose the conics reduce themselves to circles, or write

$$U = x^2 + y^2 - R^2 = 0$$
$$V = (x-a)^2 + y^2 - r^2 = 0\,;$$

R is of course the radius of the circumscribed circle, r the radius of the inscribed circle, and a the distance between the centres. Then

$$\xi U + V = (\xi+1,\ \xi+1,\ -\xi R^2 - r^2 + a^2,\ 0,\ -a,\ 0)(x, y, 1)^2,$$

and the discriminant is therefore

$$-(\xi+1)^2(\xi R^2 + r^2 - a^2) - (\xi+1)a^2$$
$$= -(1+\xi)\left(r^2 + \xi(r^2 + R^2 - a^2) + \xi^2 R^2\right).$$

Hence, *theorem*—

"The condition that there may be inscribed in the circle $x^2+y^2-R^2=0$ an infinity of n-gons circumscribed about the circle $(x-a)^2+y^2-r^2=0$, is that the coefficient of ξ^{n-1} in the development in ascending powers of ξ of

$$\sqrt{(1+\xi)\left(r^2+\xi(r^2+R^2-a^2)+\xi^2R^2\right)}$$

may vanish."

Now $$(A+B\xi+C\xi^2)^{\frac{1}{2}} =$$
$$\sqrt{A}\left\{1+\frac{1}{2}B\frac{\xi}{A}+\left(\frac{1}{2}AC-\frac{1}{8}B^2\right)\frac{\xi^2}{A^2}+\dots\right.$$

or the quantity to be considered is the coefficient of ξ^{n-1} in

$$\left(1+\frac{1}{2}\xi-\frac{1}{8}\xi^2\dots\right)\left\{1+\frac{1}{2}B\frac{\xi}{A}+\left(\frac{1}{2}AC-\frac{1}{8}B^2\right)\frac{\xi^2}{A^2}+\dots\right\},$$

where, of course,

$$A = r^2,\quad B = r^2 + R^2 - a^2,\quad C = R^2.$$

In particular, in the case of a triangle we have, equating to zero the coefficient of ξ^2,

$$(A-B)^2 - 4AC = 0\,;$$

or substituting

$$(a^2-R^2)^2 - 4r^2R^2 = 0,$$

that is,

$$(a^2-R^2+2Rr)(a^2-R^2-2Rr) = 0,$$

the factor which corresponds to the proper geometrical solution of the question being

$$a^2 - R^2 + 2Rr = 0,$$

Euler's well known relation between the radii of the circles inscribed and circumscribed in and about a triangle, and the distance between the centres. I shall not now discuss the meaning of the other factor, or attempt to verify the formulæ which have been given by Fuss, Steiner and Richelot, for the case of a polygon of 4, 5, 6, 7, 8, 9, 12, and 16 sides. See Steiner, *Crelle*, vol. ii. p. 289; Jacobi, vol. iii. p. 376; Richelot, vol. v. p. 250; and vol. xxxviii. p. 353.

2 Stone Buildings, July 9, 1853.