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Poncelet's closure theorem

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Summary. In this article we first discuss the early history of Poncelet's closure theorem. We then give a modern formulation of the theorem and we compare its modern proof with the proofs given by Poncelet (1822) and Jacobi (1828). We add a number of mathematical remarks inspired by the early proofs of the theorem.

Contents

1. Introduction

1.1 In 1822 Jean-Victor Poncelet (1788–1867) published his *Traité sur les propriétés projectives des figures.* One of the theorems in this very rich book has become known as *Poncelet's closure theorem.* It concerns conics and interscribed polygons. Let (see Figure 1.1) K be polygon with n sides $L_0, L_1, \ldots, L_{n-1}$, and vertices $P_0 = L_{n-1} \cap L_0$, $P_1 = L_0 \cap L_1, \ldots, P_{n-1} = L_{n-2} \cap L_{n-1}$. We shall call K an "interscribed polygon" of two conics C and D if all P_i are on C and all L_i are tangent to D. [There are exceptional cases in which we shall call the interscribed polygon "trivial". They occur for $n = 2k+1$ odd, if L_k is a common tangent of C and D, and, for $n = 2k$ even, if P_k is a common point of C and D. In both cases $P_j = P_{n-j}$, $0 \le j \le n$; the polygon is, as it were, folded flat (for an extensive discussion of these cases see Sections 7.6–9).] With these concepts we can formulate:

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Theorem 1.1, *Poncelet's closure theorem. Let C and D be smooth conics. If there is a non-trivial interscribed n-gon between C and D, then for every point P on C there is an interscribed n-gon which has P as one of its vertices.*

In order to explain the term "closure" in the name of the theorem we introduce some additional concepts and notation. Consider the following construction (Figure 1.2). Let P_0 be a point on C; let one of the tangents through P_0 to D be called L_0. Call P_1 the second intersection of L_0 with C. Call L_1 the second tangent to D through P_1; P_2 the second intersection of L_1 with C, etc. Repeating this construction n times we get what we shall call a "Poncelet traverse" between C and D starting in P_0, that is, a series of tangents to D such that successive tangents intersect on C. In general there are, for each point P_0 on C, two Poncelet traverses starting in P_0. If it happens in the construction that $P_n = P_0$ for some n, we say that the traverse "closes"; we have then found an interscribed n-gon between C and D. Poncelet's closure theorem says that if a Poncelet traverse starting at P_0 on C closes non-trivially after n steps, then, starting from any point on C the traverse will close after n steps.

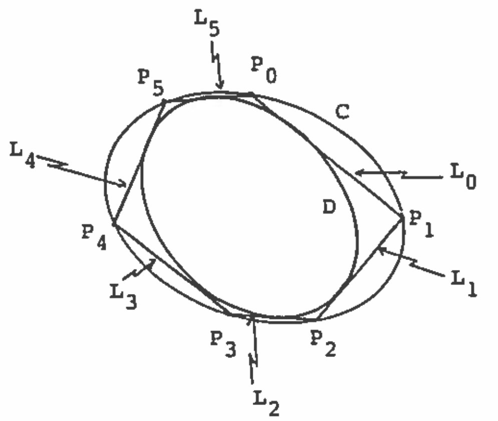


Figure 1.1

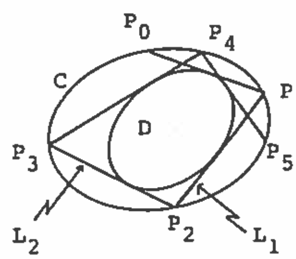


Figure 1.2

For future reference we shall also need the concept of an "almost interscribed n-gon" between C and D starting in P_0. This is an n-gon with vertices $P_0, \ldots, P_{n-1}$ on C whose first $n-1$ sides $L_0, \ldots, L_{n-2}$ are all tangent to D. Clearly, for every P on C there are in general two almost interscribed n-gons starting in P; they are constructed by making a Poncelet traverse $L_0, \ldots, L_{n-2}$, starting at P and taking $L_{n-1} = P_0$. If we consider for each P on C the nth side $L_{n-1}(P)$ of an almost interscribed n-gon between C and D starting in P, we get a family of lines. The envelope X of this family will play an important role in the proofs of the closure theorem. In the case of closure, X coincides with D.

The eighteenth- and nineteenth-century authors whose work we will discuss, used the imagery of motion in describing the concepts introduced above; thus Poncelet let P "move" along C, by which the chord L_n also moves and envelopes the curve X. In particular in the case of closure, P may be moved along C, and all the sides L_i of the interscribed n-gon will "roll" (as Poncelet said) along D, while the n-gon remains closed.

1.2 Poncelet's closure theorem has inspired many mathematicians to further study; especially from the period until the early twentieth century there is an enormous literature concerning variants, alternative proofs and generalisations of the theorem (this literature can be explored through e.g. [Dingeldey 1903] pp. 46–52, and [Loria 1889]). Then for some time the subject seems to have attracted less attention, but recently Griffiths has interpreted the theorem in a new way with the help of the theory of elliptic curves [Griffiths 1976].

It is impossible for us in one article to survey the complicated history of these studies and to do justice to all mathematicians who have contributed to the understanding and the generalization of Poncelet's closure theorem. We do not aim at such a general survey; rather we discuss a number of special topics related to the history and the modern understanding of the theorem. To be precise, we will present a modern proof along the lines suggested by Griffiths, and compare it with the proofs that Poncelet and Jacobi gave of the closure theorem. Poncelet's proof is little known and both his and Jacobi's approaches to the problem suggest a number of non-trivial questions which are of interest to the modern algebraic geometer. We will also sketch the "prehistory" of the closure theorem and call attention to some aspects of that prehistory which seem to have been insufficiently dealt with in earlier historical literature.

2. The prehistory of Poncelet's theorem

2.1 The prehistory of Poncelet's theorem is connected with a special formula from the geometry of the triangle. Let ABC be a triangle with sides of length x, y, z, circumscribed circle C with radius R, and inscribed circle D with radius r. Let a be the distance of the centres of C and D, then we have

$$a^2 = R^2 - 2rR. \tag{2.1}$$

The formula can be proved by straightforward, though somewhat complicated calculation.

The important feature of this formula is that it strongly suggests the Poncelet closure theorem for triangles interscribed between circles. Both R and r are functions of the sides x, y, z of the triangle, say

$$R = R(x, y, z) \quad \text{and} \quad r = r(x, y, z). \tag{2.2}$$

If we now fix R, r and thereby $a = (R^2 - 2rR)^{1/2}$ in such a way that there is at least one interscribed triangle between the circles C and D, then the Equations

2.2 determine a one-dimensional manifold of x, y, z-values out of which a one-dimensional family of triangles can be formed. Each of these triangles has a circumscribed circle of radius R, an inscribed circle of radius r and the distance of the centres of these circles is a. Hence each triangle is congruent to an interscribed triangle of C and D. There is, therefore, a one-dimensional family of interscribed triangles between C and D. Apart from an investigation of the limits between which the values of x, y, z will lie (in other words, whether all chords of C that are tangent to D will occur as x in the family of solutions) we have here the Poncelet closure theorem for triangles between circles.

The argument above starts from the remark that Equations 2.2 leave one degree of freedom for the x, y, z values. Eighteenth- and nineteenth-century mathematicians were familiar with such insights. We will see that during the prehistory of the Poncelet theorem some mathematicians made this remark explicitly and formulated the closure theorem as a consequence. However, the proofs they gave were insufficient.

We will now give a short sketch of the prehistory of the closure theorem. We will give some more detail about the work of Chapple and Lhuillier, because the information in the standard literature about their contributions is uninformative or unreliable.

Chapple

2.2 Formula 2.1,

$$a^2 = R^2 - 2rR,$$

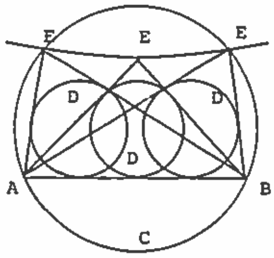
is sometimes called "Chapple's formula" after William Chapple (1717?–1781) who gave the formula in 1746 in an article in the English periodical *Miscellanea curiosa mathematica* [Chapple, 1746]. No earlier appearance of the formula is known. Most probably Chapple's work remained unknown to any larger mathematical public. Only in 1887 Mackay [1887] called attention to Chapple's article, and Cantor took over this information in his *Vorlesungen* ([1907] vol. 3, pp. 552–3). Since then the formula is attributed to Chapple.

What seems not to have been noticed is that Chapple also formulated a closure theorem and that indeed his proof of Formula 2.1 is based on the assumption (although insufficiently proved) of that closure theorem.

It is not easy to summarise Chapple's article because almost all the logic in it is wrong. Apparently Chapple was an enthusiastic amateur of mathematics, but he was, even by the standards of his own time, very weak in formal mathematical argument.

Chapple considered two circles, the one lying inside the other, with radii r and R. He noted that if the circles are concentric and $2r = R$, there is an interscribed triangle (namely the equilateral one). He argued that if $2r>R$ there cannot be an interscribed triangle but that for $2r\leq R$ the smaller circle can be placed within the larger one in such a way that there is an interscribed triangle. His algebraic arguments for these statements are muddled and insufficient. However, in the context of another argument he noted (see Figure 2.1) that, in the case $2r<R$, one may draw a chord AB in the larger circle C and a circle D with radius r touching the chord, such that, completing the circumscribed triangle ABE, the vertex E is inside C. If we now move D along AB, the vertex E will describe a curve as in the figure. We see that there are two positions for D such that E is on C, that is, such that there is an interscribed triangle. Note that this procedure already suggests that two circles may have many interscribed triangles because we may start from any chord AB between certain limits.

Figure 2.1



Indeed Chapple claimed that there are infinitely many interscribed triangles but he did not conclude this on the basis of the figure discussed above; he gave an algebraic argument. He noted that if A is the area of the triangle, A, R and r are related as

$$A=\frac{r(x+y+z)}{2}=\frac{xyz}{4R},$$

where x, y, z are the sides of the triangle (these relations were known at that time.) Chapple wrote

$$2rR=\frac{xyz}{x+y+z},$$

and concluded that for fixed r and R there are "innumerable" values x, y, z satisfying that equation, and hence innumerable triangles whose inscribed and circumscribed circles have radii r and R respectively. Chapple failed to note that there is one more relation which r, R, x, y, and z have to satisfy, namely (for instance)

$$sr^2=(s-x)(s-y)(s-z)$$

with $s=(x+y+z)/2$.

Now Chapple considered the relative positions of the inscribed and circumscribed circles and claimed that, if r and R are fixed, the distance a between the centres of these circles is also fixed. To prove this he considered an interscribed triangle between two circles and argued as follows (see Figure 2.1). Suppose that the distance is not fixed, then the inner circle could move, still allowing an interscribed triangle. Consider a chord parallel to the direction of that motion. Complete an interscribed triangle with this chord as one of its sides. (Here Chapple assumed that such a triangle exists, that is, he assumed the closure theorem, his argument is therefore circular.) The inner circle is then in the position of the left or the right circle D in Figure 2.1. But if the inner circle moves in the direction of the chord AB, the vertex E will move along the curve and hence leave the outer circle. The inner circle therefore cannot move and still admit an interscribed triangle; the distance between the circles must therefore be fixed. – The passage is a good example of Chapple's style of argument.

We quote Chapple's two propositions (V and VI) that together imply the closure theorem:

> "An infinite number of triangles may be drawn, which shall inscribe and circumscribe the same two circles; provided their diameters, with respect to each other, be limited, as in the two last propositions [i.e. provided $2r \leq R$]. The nearest distance of the peripheries of two given circles, or, which amounts to the same, the distance of their centres, in order to render it possible to inscribe and circumscribe triangles, is fixed, and will always be the same." ([Chapple 1746], pp. 119–120.)

Having come so far, Chapple used the closure theorem to derive a formula for a in terms of R and r. He argued that if there is an interscribed triangle, there are innumerable such triangles, hence one may calculate a in the special case that the triangle is isosceles and has its line of symmetry through the centres of the circles. In that case the calculation is indeed straightforward, and Chapple found

$$a^2 = R^2 - 2rR,$$

the correct formula.

So far Chapple. We will note that several of his ideas occurred again in the prehistory of Poncelet's theorem, so we may conclude that, despite his failures in the realm of logic, Chapple had grasped some essential aspects of the problem.

Euler

2.3 Some nineteenth-century authors attributed the formula

$$a^2 = R^2 - 2rR$$

to Euler. This was probably done because Euler had studied, in a famous paper [Euler, 1765], the positions and mutual distances of four special points of a triangle: the centre of gravity, the intersection of the altitudes, the centre of the inscribed circle and the centre of the circumscribed circle. For the distance between the last two points Euler gave the formula

$$a^2 = \frac{(xyz)^2}{16 \cdot A^2} - \frac{xyz}{x+y+z}$$

in which x, y, z are the sides of the triangles and A its area. This formula did not express a in terms of r and R, nor did Euler investigate the relation between a, r and R in his article. Hence he was not lead to a formula from which a closure theorem could be detected. We have not found the Formula 2.1 in other works of Euler. (Of course, it may well be that the formula occurs in Euler's unpublished manuscripts; about this, perhaps [Belyj, 1983] gives more information, but we have not been able to consult that article.) Fuss, who in 1797 published the formula and who was certainly well acquainted with Euler's work, did not attribute it to Euler. It seems, therefore, that the nineteenth-century attribution of the formula to Euler is wrong.

Fuss

2.4 In a paper of 1797 [Fuss 1797], Nicolaus Fuss studied quadrangles that admit an inscribed and a circumscribed circle. He derived for such quadrangles a relation between r, R and a, namely

$$(R^2 - a^2)^2 = 2r^2(R^2 + a^2).$$

He proved the relation by direct calculation; that calculation turned out to be quite involved. At the end of his article he noted and proved that also in the case of the triangle there is such a relation, namely

$$a^2 = R^2 - 2rR.$$

This seems to be the first occurrence of the formula in a source that was accessible and known to the international mathematical public.

Fuss did not discuss closure properties of interscribed triangles or polygons. In a sequel article [Fuss 1802] he studied 5-, 6-, 7-, and 8-gons admitting both inscribed and circumscribed circles. He reported that he had not been able to calculate in general the relation between a, r and R in these cases. Instead he dealt with the special cases in which the polygons are symmetrical with respect to the line through the centres of the inscribed and circumscribed circles. Here he was able to derive the required formulas (cf. Section 2.7). As Jacobi was to notice later (see Section 6.4), the closure theorem for interscribed n-gons between circles implies that Fuss's results, derived from the special case of symmetrical polygons, are indeed general. Fuss himself, however, did not notice this, so that we may conclude that he had not seen a connection of his formulas with a closure theorem.

Repeated rediscovery — Lhuillier and others

2.5 Although Fuss had published his results in an internationally prestigious journal, several mathematicians failed to notice them and found Formula 2.1 independently. This appears from the first volume of Gergonne's *Annales de mathématiques pures et appliqués* of 1810–1811. On pp. 62–64 [Gergonne 1810] we find two problems proposed by the editors (probably by Gergonne). They are:

1. Given a circle C and a point P inside it, to prove that there is exactly one length such that a circle D around P with that length as radius, admits a circumscribed triangle which is also an inscribed triangle of C.
2. Given a circle D and a point P, to prove that there is exactly one length such that a circle C around P with that length as radius admits an inscribed triangle which is also a circumscribed triangle of D.

Clearly the proposer of these problems knew the formula

$$a^2 = R^2 - 2rR$$

and had noted that, given a and r (or a and R) there is exactly one R (or r) which satisfies the formula. This is what he suggested his readers prove.

The problems were solved by Lhuillier [1810]. Lhuillier started with a proof of the formula and the editors added a footnote explaining that the same result had been sent to them by a Monsieur Kramp, professor at Strassbourg, and that it had been earlier communicated to them by the late Monsieur Mahieu, professor at the Collège of Alais, who himself had learned it from a Monsieur Maisonneuve, "ingénieur des mines". They added Maisonneuve's proof.

Lhuillier's answer to the problem is interesting because, after proving the formula, he did indeed conclude a closure theorem from it. He formulated it as follows:

> "There is therefore an unlimited number of triangles that can be at the same time inscribed in one circle and circumscribed around another circle, when the radii of these circles and the distance of their centres are related by the equation $a^2 = R(R-2r)$." ([Lhuillier, 1810] p. 155; our translation)

This is a fairly clear enunciation of the closure theorem. Lhuillier's proof consisted merely in stating that, if a, r and R satisfy Equation 2.1, then a triangle which is circumscribed around D and whose one side is a chord of C, will necessarily be inscribed in C. No further argument was given; the proof is therefore insufficient. But we may conclude that Lhuillier had hit upon the idea of a closure theorem by noting the fact that the relation

$$a^2 = R^2 - 2rR$$

leaves one degree of freedom for the choice of the sides of the triangle.

Poncelet

2.6 As we shall report in more detail below (Section 4.5), Poncelet found and proved the closure theorem in 1813–14, and published it in 1822. However, in 1817 he published an article [Poncelet 1817] in Gergonne's *Annales* in which he announced new methods in geometry. As an example of a problem solvable by these methods he proposed the construction of an interscribed n-gon for two given conics. Poncelet may have expected that his readers would hit on the closure theorem while dealing with this problem. Apparently no one did, and indeed the formulation of the problem did not at all point towards a closure property, it rather suggested that for any pair of conics there are interscribed n-gons.

Poncelet himself did not notice the relation of his closure theorem to Formula 2.1. and the related studies mentioned above; it was Jacobi who first pointed out that connection (cf. Section 6.4).

Steiner

2.7 There are two notes by Steiner [1827] which, although published after Poncelet's *Traité*, belong to the prehistory of Poncelet's theorem. The first is a list of problems and theorems, published in the second volume of Crelle's *Journal für die reine und angewandte Mathematik*. Steiner left it to the reader to solve the problems and prove the theorems. Problem 3 is:

> "If a given (irregular) polygon (n-gon) is such that circles can be drawn in and around it, it is required to find a relation between the radii (r, R) of both circles and the distance (a) of their centres. (It is well known that for the triangle, the equation, first found by Euler, is $a^2 = R^2 - 2rR$.)" ([Steiner 1827] p. 96; our translation)

(On the attribution to Euler, see Section 2.3; apparently Steiner at that time did not know about Fuss's work.) In a later note in the same volume Steiner gave the required equations for $n = 3, 4, 5, 6, 8$, namely

$$n=3 \quad R^2 - a^2 = 2rR$$

$$n=4 \quad (R^2-a^2)^2 = 2r^2(R^2+a^2)$$

$$n=5 \quad r(R-a) = (R+a)[(R-r+a)(R-r-a)]^{1/2} + (R+a)[(R-r-a)\,2R]^{1/2}$$

$$n=6 \quad 3(R^2-a^2)^4 = 4r^2(R^2+a^2)(R^2-a^2)^2 + 16r^4a^2R^2$$

$$\begin{aligned} n=8 \quad & 8r^2[(R^2-a^2)^2 - r^2(R^2+a^2)\times \\ & \times\{(R^2+a^2)[(R^2-a^2)^4+4r^4a^2R^2] - 8r^2a^2R^2(R^2-a^2)^2\} \\ & \quad = [(R^2-a^2)^4 - 4r^4a^2R^2]^2. \end{aligned}$$

But he did not give proofs, nor did he explicitly refer to a closure property.

3. Poncelet's approach to projective geometry

3.1 Poncelet formulated the closure theorem in Sections 565–567 of the *Traité* [1822]. His proof of the theorem was based on many different results scattered through the preceding paragraphs. We will give a sketch of that proof in Section 4, but before doing so we have to discuss two special concepts that were crucial to Poncelet's set-up of projective geometry, and that gave his work a character which may seem rather strange to the modern mathematician. These concepts are the "principle of continuity" and the "ideal chords".
In his *Traité* Poncelet adopted a strongly "synthetic" approach to projective geometry, avoiding analytical techniques as much as possible. This style marked the final stage of his development as a geometer; in the beginning of his research Poncelet used analytical techniques freely. In fact many of the results in the *Traité* (including the closure theorem, see Section 4.5) were first proved analytically.

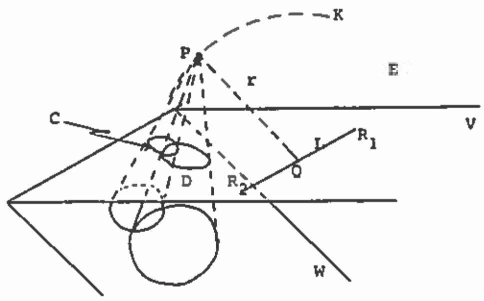
Poncelet found these early results while in captivity as a prisoner of war at Saratov on the Wolga from winter 1812/13 till June 1814. Back in France, in the period 1815–1822, he reworked the results. Later in life he decided to publish the notebooks, both from the period in Saratov and from the later years before the publication of the *Traité*. They appeared in 1862 and 1864 respectively [Poncelet 1862]. The notebooks from Saratov show Poncelet using analytical techniques without seriously questioning the occurrence of "imaginary" and "infinite" quantities in the formulas. The term "imaginary" should be taken literally here: for mathematicians of Poncelet's age the term referred to quantities which do not exist but are "imagined" in order not to obstruct the smooth progress of the analytical operations. Mathematicians had become convinced that all such "imaginary" quantities can be written in the form $a+b\sqrt{-1}$, with a and b real. The geometrical interpretation of these quantities as forming a plane in which the real line is embedded, emerged in the early nineteenth century (in studies by Gauss, Wessel and Argand), but mathematicians took a long time to accept it. It is not clear whether Poncelet was aware of this interpretation and, if so, when he learned about it; the conception of the complex plane occurs neither in his early analytical studies nor in his later synthetic work.

In his early analytic studies Poncelet had been confronted with imaginary elements in connection with two major issues: the intersection of two conics or a conic and a line, and the projective images of pairs of conics. He wanted to derive properties of pairs of conics by projectively generalising properties of pairs of circles. To do so he needed the following theorem:

Theorem 3.1, *Projection theorem. Any pair* (C, D) *of conics in a (real) plane* V *is the projective image of a pair of circles.*

Let (see Figure 3.1) V be embedded in the real space E. Poncelet then had to find a point P and a plane W in E such that the perspective projection $V \rightarrow W$ with centre P maps both C and D on circles. In the case that C and D have

Figure 3.1



no more than two points of intersection this is indeed possible and Poncelet showed how such P and W can be located in space. He proceeded as follows: He showed that C and D determine a certain line L in V and two points R_1 and R_2 on L. The middle of $R_1 R_2$ is called Q. Poncelet considered a circle K in E with center Q and lying in a plane perpendicular to L. The radius r of K is equal to $(R_1 R_2)/2$. Poncelet then proved that any point P on K and not in V, and any plane W parallel to the plane through P and L can serve as the centre and plane respectively of the required projection. Poncelet proved this result analytically while in Saratov ([Poncelet, 1862] vol. 1, pp. 287–307); the *Traité* contains a synthetic proof ([1822] Section 121), to which we shall return in Section 3.4.

However, in the case that C and D intersect in more than two points, the analytical calculation leads to an imaginary value for the radius r of K. Hence in that case there is no real centre for the required projection. This is in fact evident because in the case of a real projection the images of C and D would still have four intersections, so that, if they were circles, they would coincide. So Poncelet met a major obstacle in his programme of deriving properties of pairs of conics from properties of pairs of circles. He did not remove this obstacle in the way that seems obvious to the modern mathematician, namely by introducing imaginary centres of projection. In the Saratov period he struggled with the obstacle but did not come further than expressing the conviction that, although in the case of four intersections a pair of conics could not actually be the projective image of a pair of circles, nevertheless it could be *considered* as such. Moreover, he had noticed that in the case that C and D intersect in more than two points, the points R_1 and R_2 used in the construction explained above, coincide with the two additional points of intersection. From this insight he developed the idea that in the case of two or less intersections the points R_1 and R_2 could be somehow *considered* as the two missing points of intersection of C and D; the segment $R_1 R_2$ would be in a sense a "common chord" of C and D, and Q would be the middle of such a chord.

Back in France after his captivity, Poncelet found a lively discussion among mathematicians about the acceptability of various imaginary quantities in mathematics; not only the reliability of calculations involving $\sqrt{-1}$ was questioned, also negative quantities, infinitely distant points, and infinitely small or large quantities were debated. Confronted with these discussions Poncelet set himself the task of developing a science of geometry which concerned real space, which was independent from analytical techniques and which still admitted the generality of arguments and results which the analytical approach attained by the use of imaginary quantities. To reach that goal Poncelet started from the ideas gained earlier in connection with the projection theorem, and developed them into the concept of ideal chords and the principle of continuity. By help of these he was able to reformulate his earlier results and arguments and to present them in the *Traité* in the synthetic but general style he envisaged for geometry.

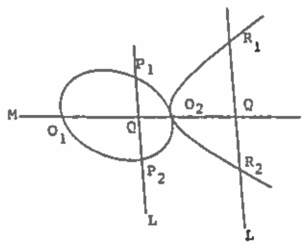
We shall now explain the concept of ideal chord and the principle of continuity as far as is necessary for the understanding of Poncelet's proof of the closure theorem in the *Traité*.

Ideal chords

3.2 Poncelet's geometry concerned the *real* plane and the *real* space, extended with elements in infinity. The plane (space) was taken as a primitive, further undefined object; it was not considered in the modern sense as the set of ordered pairs (triplets) of real numbers. Poncelet did use real numbers, or rather real magnitudes, ad hoc when he measured distances, but these magnitudes did not serve as the foundation of the geometrical objects. In the *Traité* Poncelet took the existence of infinite elements for granted. In earlier studies, however, he had shown that their adoption can be justified by the principle of continuity (cf. [Poncelet 1819], esp. pp. 344–45).

Poncelet did not extend the plane by adding complex points or lines. However, by means of his concept of "ideal chords" of conics ([1822] Sections 54–55) he was able to develop a machinery for the study of conics which is to a large degree equivalent to the extension of the real affine plane to the complex affine plane. Let C be a (real) conic and L a line. If L intersects C in points P_1 and P_2, then P_1P_2 is the chord of C corresponding to L. If L does not intersect C, Poncelet constructed a linesegment R_1R_2 on L which he called the "ideal chord", as follows (see Figure 3.2):

Figure 3.2



Let M be the diameter of C which is conjugate to the direction of L. M intersects C in O_1 and O_2. Consider chords parallel to L, intersecting C in P_1 and P_2 and M in Q. We then have

$$(QP_1)^2=\gamma(O_1Q)(O_2Q)$$

for some constant γ. Poncelet extended this relation to the case that L lies outside C. He considered points R_1 and R_2 on lines parallel to L, such that the middle Q of $R_1 R_2$ is on M and

$$(QR_1)^2 = \gamma(O_1 Q)(O_2 Q)$$

These points form a new conic which Poncelet called the supplementary conic of C; each direction defines a different supplementary conic. Poncelet now called the segment $R_1 R_2$ the "ideal chord" cut off by L from C. By this construction every line cuts off either a real or an "ideal" chord in C. Note that the segment $R_1 R_2$ is not "imaginary" but well defined in the real plane; what is "ideal" about it is that it should be a chord of C. Poncelet made a sharp distinction between imaginary and ideal elements in his geometry, "imaginary" are those points and lines which do not exist in the real plane but are *imagined* to exist in order not to block the argument; "ideal" are those elements which do exist but are imagined to have a certain role or property which in reality they don't have.

It is easily seen that the following construction, using the complex extension of the affine real plane, yields the same $R_1 R_2$: Let T_1 and T_2 be the complex points of intersection of C and L; let S be the middle of $T_1 T_2$. Because T_1 and T_2 are conjugate, S is real. In fact, S is the intersection of L with its conjugate diameter M, so S coincides with Q. Hence

$$T_i = S \pm iD$$

for some real distance D. Then

$$R_1 = S + D \quad \text{and} \quad R_2 = S - D$$

are the real endpoints of Poncelet's "ideal chord". We see from this construction that Poncelet's ideal chord determines the complex points of intersection of C and L and vice versa. This explains that Poncelet was able by means of this concept to work out a theory of conics incorporating many features that nowadays can be introduced only by the complex extension of the real plane. In particular Poncelet showed (Sections 58–59) that any two (real) conics have two "common chords", that is, two (real) segments $I_1 I_2$ and $I_3 I_4$ which are real or ideal chords to both conics. Indeed the points $I_1, \ldots, I_4$ correspond to the four complex or real points of intersection of the two conics, and if two points of intersection are non-real (and conjugate), they yield an ideal chord.

Poncelet was aware that the endpoints of the chords, and the chords themselves, may lie on the line in infinity. He noted in particular that there are two imaginary points on the line at infinity, through which all circles pass (Section 94); in this way he introduced the later so-called "circular points".

The "principle of continuity"

3.3 Poncelet developed his "principle of continuity" in order to overcome obstacles as the one explained above in connection with the projection theorem. Poncelet could prove that theorem for certain positions of the conics but not for others. His principle enabled him nevertheless to pronounce the theorem as general.

Poncelet discussed the principle and its use at great length in various places in his *Traité* ([1822], notably Introd. pp. xiii–xiv and Sections 135–140) and he formulated it in various ways. It may be summarized as follows. Suppose one considers a figure in the plane and derives certain properties from certain data about the figure and from certain general theorems. Suppose also that the figure can be continuously deformed in such a way that the data and the theorems remain valid. Then the derived properties also remain valid, even if during the transformation certain magnitudes change of sign or vanish, and also if, during the deformation, certain aspects of the figure that were used in deriving the properties, do no longer apply. Moreover it will be a priori predictable in which positions of the figure the signs change or the magnitudes disappear (cf. [1822], Inrod. p. xiii, cf. also our proof of Lemma 8.5 and the Remark 8.8).

In the *Traité* Poncelet primarily considered transformations induced by projecting one figure F from continuously varying centres onto continuously varying planes. With help of such projections he could generalize properties proved for the single figure F to properties valid for all the projective images of F. However, there are also continuous transformations that cannot be produced in this way by real projections but which still apparently leave the relevant properties invariant. Such transformations are for instance those which transform a pair of conics from a situation with two intersections to a situation with four intersections. In such cases Poncelet invoked the principle of continuity to assert that properties that can be proved to be invariant under transformations induced by real projections, remain invariant also under the other type of transformation.

Thus the principle of continuity enabled Poncelet to avoid the use of projections with imaginary centres or imaginary planes of projection. In other words, the principle of continuity served a similar function as the "ideal chords"; it provided a powerful means to derive results which nowadays can be proved, or even formulated, only by embedding the real projective plane (space) in the complex one.

3.4 We shall clarify Poncelet's use of the principle of continuity by discussing his proofs of three theorems which he later used in the proof of the closure theorem; one of them is the projection theorem discussed above. The theorems are:

Theorem 3.4.1 *([1822], Sections 109–111). Let C be a conic and L a straight line. Then C and L are the projective images of a circle and the line at infinity.*

Proof. Call the plane of C: V and embed it in a real space E. If L does not intersect C it is possible, by elementary theory of sections of cones, to find a point P and a plane W in E such that C is projected from P onto a circle in W and such that the plane through P and L is parallel to W. The projection $V \to W$ from P then maps L onto the line at infinity. However, if L intersects C, such P and W cannot be found in the real space E. Still Poncelet claimed, on the basis of the principle of continuity, that the theorem should be considered as valid in general.

Theorem 3.4.2. *Projection theorem ([1822], Section 121). Let C_1 and C_2 be two conics. Then the pair C_1, C_2 is the projective image of a pair of circles.*

Proof. Suppose C_1 and C_2 have not more than two real points of intersection. Then they also have an "ideal common chord" along a certain real line L. By Theorem 3.4.1 there is a projection which maps C_1 onto a circle $\underline{C}_1$ and L onto the line at infinity L_0. The image $\underline{C}_2$ of C_2 passes through the points in which L_0 intersects $\underline{C}_1$. These are the circular points, hence $\underline{C}_2$ is also a circle. By the principle of continuity the theorem should be considered to apply in general, that is, also in the case that C_1 and C_2 have four real points of intersection.

Theorem 3.4.3 *([1822], Section 131). Let C_1 and C_2 be two conics, which are tangent to each other in two different points. Then the pair C_1, C_2 is the projective image of a pair of concentric circles.*

Proof. By Theorem 3.4.2 the two conics are the projective images of circles. The two points of contact are the images of the two circular points. Hence the two circles are tangent to each other in the circular points, and the pole of the line in infinity is the same for both circles; therefore they are concentric.

The three theorems are indeed correct if we allow the complex extension of the affine plane. Hence Poncelet's principle of continuity led him to correct results. It should be noted that, in the context of real geometry, the three theorems are successively more counterintuitive. The first two theorems can still be "seen" to be correct in certain cases, but the third does not apply at all in the real case; if the two tangent conics were images of real circles, these circles would be tangent in two points as well, hence they would coincide. This shows how daring Poncelet's use of ideal chords and of the principle of continuity really was. Poncelet was well aware of that; it may be illustrative to quote here one of the many assertive sentences he wrote in defence of the use of these new concepts:

> "Is it not at least as necessary to teach the resources used at different times, by men of genius, to arrive at the truth, than to teach the laborious efforts which they subsequently had to make to prove their results according to the taste of minds that were timid or unable to place themselves on their level." ([1822], Intr. p. xiv; our translation)

3.5 Before turning to Poncelet's proof of the closure theorem we make a final remark about his approach to projective geometry.

Poncelet proposed to openly use the principle of continuity in geometry. He also claimed that, in fact, the principle was already used tacitly in almost all branches of mathematics. In algebra, negative and imaginary quantities were justified by arguments akin to the principle, and in analysis infinitely small and infinitely large quantities were justified in a similar way. Poncelet was right; the foundational questions about the concepts of imaginary and infinitesimal magnitudes and numbers were still very much open in his time, and in using these concepts mathematicians often relied on the tacit assumption that the usual mathematical laws also apply to these entities.

However, the foundational questions were recognized as important and Poncelet's conscious adoption of the principle of continuity may be seen as fitting in one particular style of reaction to these questions. Poncelet was not alone in his approach; in particular the "principle of the permanence of forms", which English algebraists of that period used to justify calculations with negative and complex quantities, was very similar to Poncelet's principle of continuity.

Ultimately, nineteenth-century mathematics found a convincing answer to the foundational questions, but not in the manner envisaged by Poncelet. The foundations, especially of analysis, were secured by the installation of rigour through arithmetization, an approach for which the works of Cauchy and Weierstrass became paradigmatic. That approach is still so familiar to the modern mathematician that to him Poncelet's style may seem unacceptably unfounded and dangerous.

It seems to us that such a judgement is too rash. The programme of rigorization through arithmetization found the solution of the foundational problems in precise definition of the objects of mathematical study; the complex numbers were interpreted as pairs of real numbers, and the real numbers themselves were explicitly constructed from the natural numbers. Thus real spaces could be interpreted as real number spaces and they could be embedded in explicitly constructed complex spaces. In Poncelet's approach we see an attempt to solve the foundational problems in an entirely different way. Rather than extending or embedding the objects of mathematics (real numbers, real spaces), Poncelet wanted to extend the mathematical rules of inference. For that purpose he introduced ideal elements and applied the principle of continuity. By the time Poncelet worked out this approach it could not be clear to mathematicians that the solution of the foundational problems lay in the one rather than in the other direction. Hence Poncelet's approach seems a justifiable enterprise.

Moreover, modern insights in logic and model theory would make us aware that extension of properties or rules of interference and extension of objects or models are in a way interchangeable. With this new hindsight, Poncelet's approach would the more seem sensible and valuable in its own right.

4. Poncelet's proof of the closure theorem

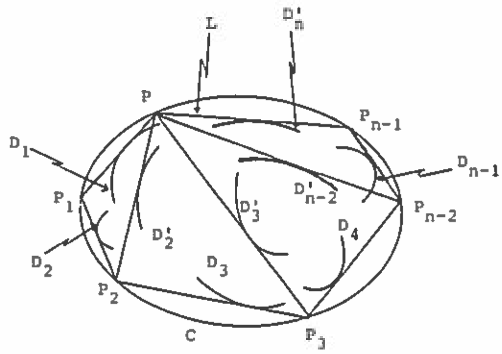
Poncelet's general theorem

4.1 Poncelet's closure theorem is in fact a corollary of a much more general theorem which Poncelet presented in Section 534 of his *Traité*, and which we shall call the "general theorem". It concerns sets of conics which, in Poncelet's terminology, "have common real or ideal chords". This means that the conics have four different points (real or complex) in common. In the following we will denote this situation by saying that the conics belong to one pencil. Poncelet's general theorem is as follows.

Theorem 4.1.1, "*general theorem*". *Let (see Figure 4.1) C, D_1, D_2, ..., D_{n-1} be conics from one pencil. Consider an inscribed n-gon of C whose first vertex is P and whose first $n-1$ sides are tangent to the successive D_i. Then if P moves along C the n-th side L of the polygon will envelop a curve which is again a conic belonging to the same pencil.*

[Poncelet also considered cases where some of the D_i or the enveloped curve are degenerate, see below in the present section. Poncelet was aware that for each P there are several such polygons, because at each vertex of a traverse starting in P there are in general two choices for the next side. Correspondingly, several curves arise as envelopes of n-th sides L. Poncelet here considered only one such envelope, but in the case $n=3$ he discussed all the different possibilities, see Section 4.4.]

Figure 4.1



For future reference we will refer to the special case $n=3$ as the *main theorem*; thus

Theorem 4.1.2, "*Main Theorem*". *The general theorem applies in the case $n=3$.*

Proof (of the general theorem from the main theorem). Poncelet first proved the main theorem. This is the hardest and the most interesting part and it will be separately discussed in Section 4.3. Then, with the main theorem proved, Poncelet argued for the general case as follows (Section 534). Consider the polygon split up by diagonals through P into triangles PP_1P_2, PP_2P_3, ..., $PP_{n-2}P_{n-1}$. Applying successively the main theorem we find that PP_2 envelops a conic D'_2 from the pencil; PP_3 envelops a conic D'_3 from the pencil etc. Hence, ultimately, the side PP_{n-2} of the triangle $PP_{n-2}P_{n-1}$ is tangent to a conic D'_{n-2} and $P_{n-2}P_{n-1}$ is tangent to D_{n-1}, so that the remaining n-th side $L=P_{n-1}P$ envelops a conic D'_n which belongs to the same pencil.

Poncelet was aware that for certain positions of the conics C and D_i, a diagonal may pivot around one point, or move parallel to itself, rather than envelop a curve. He considered these occurrences as limiting cases of the general situation and he discussed them in connection with the case $n=3$, see Section 4.4. [For a modern explanation of such phenomena of degeneration see the reference to the theory of "complete conics" in Section 9.9; cf. also Lemma 8.5.i for a special case, and Remark 9.11.]

Poncelet's closure theorem

4.2 If all D_i in the general theorem coincide and are equal to D we are in the situation of Poncelet's closure theorem. (Note that C and D can now be arbitrary.) Poncelet dealt with that case as follows (Sections 565–566). Suppose that, starting from P_0 on C, an interscribed n-gon K closes. Consider the family of almost interscribed n-gons starting from points on C. Their last sides will envelop, according to the general theorem, a conic D' from the same pencil as C and D. All the sides of K are tangent to D' because they occur as last sides of almost interscribed n-gons starting in any one of the vertices of K. If $n \geq 5$ it follows that D and D' have five or more tangents in common; so they coincide. That means that all almost interscribed n-gons are indeed interscribed n-gons, which proves the closure theorem. Poncelet dealt with the cases $n=3$ and $n=4$ by showing that the last side of the triangle or quadrangle starting from P_0 touches both D and D' in the same point. Hence D and D' have one point (or rather two coinciding points) in common. As they belong to the same pencil, they coincide.

It may be of interest to quote here Poncelet's own enunciation of the closure theorem:

> "If any polygon is at the same time inscribed in a conic and circumscribed about another conic, then there are an infinity of such polygons with the same property with respect to the two curves; or rather, all those polygons which one would try to describe at will, under these conditions, will close by themselves on these curves. And conversely, if it happens that, while trying to inscribe arbitrarily in a conic a polygon whose sides will touch another, this polygon does not close by itself, it would necessarily be impossible that there are others which do have that property." ([1822] Section 566; our translation)

The case $n=3$, the "main theorem"

4.3 We now come to the most interesting part of Poncelet's proof, namely the case of the triangle in the general theorem. Poncelet gave the proof in the case of circles, using the projection theorem (cf. Sections 3.1 and 3.4) to generalize to arbitrary conics. Let three conics from the same pencil be given. Any pair of them can be considered as the projective image of a pair of circles, and the third will then be the image of a conic belonging to the same pencil as the two circles, that is, again a circle. Poncelet proved:

Lemma 4.3. *Main theorem for circles. Let (see Figure 4.2) C, D_1 and D_2 be three circles from one pencil. Let PR_1R_2 be an inscribed triangle of C whose sides PR_1 and PR_2 touch D_1 and D_2 respectively. Then, if P varies along C, the sides R_1R_2 envelop a circle from the pencil. Moreover, if $Q_i (i=1, 2)$ are the points in which the chords PR_i* touch the circles *D_i, and $H=R_1Q_1 \cap R_2Q_2$, then the chord R_1R_2 touches D at the point $G=PH \cap R_1R_2$; this provides a construction of points on the envelope D. (Section 531).*

In his proof of this lemma Poncelet used a number of results which he had derived in earlier sections of the *Traité*. The total of his argument can be split up in three parts, one concerning a curve that arises in a construction involving rotating chords, one consisting in the application of certain elementary geometrical theorems in the configuration of the lemma, and the third (and most interesting) part consisting in two brilliant ad hoc arguments. We shall deal with these parts separately below.

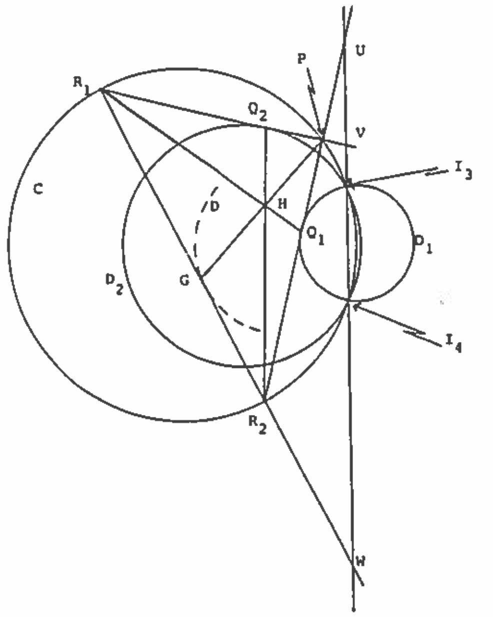


Figure 4.2

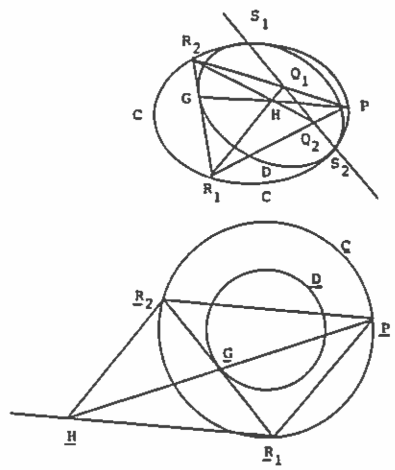


Figure 4.3

Proof of the Lemma. Part I (Sections 431–433) Let (see Figure 4.3) C be a conic and Q_1 and Q_2 two points not on C. For P on C consider the two chords PQ_1R_2 and PQ_2R_1 with R_i on C. For P variable on C, R_1R_2 will be a family of chords. Poncelet proved that this family of chords envelops a conic D which is tangent to C in the points S_1 and S_2 in which Q_1Q_2 intersects C. Moreover, in each position R_1R_2 touches D in a point G such that PG, Q_1R_1 and Q_2R_2 intersect in one point H. The proof is as follows.

By Theorem 3.4.1 above, C and the line Q_1Q_2 are the projective image of a circle $\underline{C}$ and the line at infinity. Then S_1 and S_2 are the images of the circular points I_1 and I_2. We denote the originals of the other points by underlining. Poncelet tacitly assumed that Q_1 and Q_2 are real points at infinity so that for variable $\underline{P}$ on $\underline{C}$, the chords PR_1 would be parallel, as well as the chords PR_2. [That assumption is not correct. It is easily seen that under a projective mapping either all the real points of a line have real images or at most two of them. The originals of S_1 and S_2 are not real, so there will be at most two points on $S_1 S_2$ with real originals, hence in general Q_1 and Q_2 will not be real. Therefore Poncelet's use of real parallelism and real angles in the remainder of the proof is unfounded. It is possible to prove this part of the theorem along the lines implicit in Poncelet's approach by using cross ratios to generalize the concepts of parallelism and angles from the real to the complex case. That procedure, however, is rather laborious. In Section 8.5 we will give an alternative proof by means of closed conditions on Zariski-dense sets.] Poncelet concluded that the angle between $\underline{PR}_1$ and $\underline{PR}_2$ is constant, so that $\underline{R}_1 \underline{R}_2$ is a chord of constant length in the circle $\underline{C}$. Therefore $\underline{R}_1 \underline{R}_2$ will envelop a circle D concentric with $\underline{C}$. $\underline{D}$ is the origin of D because enveloping is a projective invariant. $\underline{D}$ touches $\underline{C}$ in I_1 and I_2 (the images of S_1 and S_2). Hence D will be a conic touching C in S_1 and S_2. Moreover $\underline{G}$ is the middle of $\underline{R}_1 \underline{R}_2$, and, completing the parallellogram $\underline{PR}_1 \underline{HR}_2$, we have that $\underline{PGH}$, $\underline{R}_1 \underline{Q}_1 \underline{H}$ and $\underline{R}_2 \underline{Q}_2 \underline{H}$ are each collinear, hence PG, $R_1 Q_1$ and $R_2 Q_2$ intersect in one point H. This completes the proof of part I. – Poncelet dealt here with the same case as in Theorem 3.4–3 above; it should be noted how heavily his argument relies on the principle of continuity. For a modern version of this argument see Sections 8.3–8.5.

Proof of the Lemma, Part II (Section 531) The second part of the proof concerns the application of a number of elementary geometrical theorems to the configuration of the lemma. Consider that configuration in the case that the three circles C, D_1 and D_2 intersect in real points I_3 and I_4 (cf. Figure 4.2). We have P, R_1 and R_2 on C, PR_1 tangent to D_2 in Q_2, PR_2 tangent to D_1 in Q_1; $Q_2 R_2$ and $Q_1 R_1$ intersect in H; PH intersects $R_1 R_2$ in G. Call U, V, W the intersections of PR_2, PR_1 and $R_1 R_2$ with the line through I_3 and I_4. We have now the following relations:

$$PQ_1 \cdot R_2 G \cdot R_1 Q_2 = Q_1 R_2 \cdot GR_1 \cdot Q_2 P \tag{a}$$

$$R_2 W \cdot PU \cdot R_1 V = R_1 W \cdot R_2 U \cdot PV \tag{b}$$

$$(UQ_1)^2 = UP \cdot UR_2 \tag{c}$$

$$(VQ_2)^2 = VP \cdot VR_1 \tag{d}$$

Equation (a) is the theorem of Ceva applied in triangle $PR_1 R_2$. Equation (b) is the transversal theorem applied for the same triangle and transversal $I_3 I_4$. Equations (c) and (d) are proved by noting that, by the power theorem for circles, the terms in (c) and (d) are equal to $UI_3 \cdot UI_4$ and $VI_3 \cdot VI_4$ respectively. Poncelet concluded from (a), ..., (d) that

$$(WG)^2 = WR_2 \cdot WR_1 \tag{e}$$

He did so by referring to earlier sections (161 – 163) where he had proved that (c), (d), (e), together with (b), imply (a). Apparently he assumed that any four of these relations implies the fifth. This is in fact true; it can be checked by somewhat complicated calculations which we omit here.

Relation (e) has an important consequence, namely that $R_1 R_2$ is tangent in G to the circle through G, I_3 and I_4. This fact is used in the following part to prove that that circle is indeed the envelope of the family of chords $R_1 R_2$.

Proof of the Lemma, Part III (Section 531) Poncelet completed his proof as follows. Consider the situation of Figure 4.2. We want to determine the curve D which is the envelope of all chords $R_1 R_2$ for varying P. Consider an infinitely small displacement of P along C. This causes displacements of the chords PR_1, PR_2 and $R_1 R_2$, which, as long as the displacement of P is infinitely small, may be considered the same as the displacements that would occur if Q_1 and Q_2 were fixed and PR_1 and PR_2 rotated around them. [Poncelet gives no further argument; for a modern proof see Section 8.] In that case it follows from Part I that $R_1 R_2$ envelops a conic E, and it touches E in the point G determined as the intersection of PH with $R_1 R_2$, where H is the intersection of $R_1 Q_1$ and $R_2 Q_2$ (cf. Figures 8.2 and 8.3). Because by infinitesimal displacement the motion of $R_1 R_2$ when PR_1 and PR_2 touch D_2 and D_1, does not differ from its motion when PR_1 and PR_2 rotate around Q_2 and Q_1, we conclude that $R_1 R_2$ will touch D in G as well. We have then a construction of points on D, namely; join $R_1 R_2$, $R_1 Q_1$, $R_2 Q_2$, call H the point $R_1 Q_1 \cap R_2 Q_2$, then $PH \cap R_1 R_2$ is a point G on D. [For a proof and a generalisation of this construction to arbitrary algebraic curves C and D, see Section 8.]

For future reference we quote here the words in which Poncelet presented the arguments about infinitesimal motions (we add the original text in this case to give one example of the flavour of Poncelet's French; we have changed the letters so as to fit our Figure 4.2):

> «Pour le démontrer, commençons par rechercher le point de contact G du côté $R_1 R_2$ avec la courbe qu'il enveloppe dans ses diverses positions. J'observe d'abord que, si l'on imprime au triangle $PR_1 R_2$ un mouvement infiniment petit, ou qu'on le dérange infiniment peu de sa position actuelle, il arrivera que les côtés de ce triangle tourneront, ou tendront à tourner autour des points de contact Q_1, Q_2, G, qui leur appartiennent respectivement. Mais, en faisant abstraction, pour un moment, de la courbe qu'enveloppe en général le côté $R_1 R_2$, il est visible que ce côté tendra aussi à rouler (431) autour d'une section conique ayant un double contact avec le cercle C; donc le point de contact de cette section conique et du côté $R_1 R_2$ est aussi celui de ce même côté avec la courbe inconnue; et par conséquent, si l'on trace les droites $R_1 Q_1$, $R_2 Q_2$, et qu'on joigne le point H de leur croisement avec le sommet P, par la droite PH, sa direction indéfinie ira rencontrer celle de $R_1 R_2$ au point de contact G dont il s'agit (433).» ([1822], Section 531)

"To prove this, let us begin by searching the point of contact G of the side $R_1 R_2$ with the curve that it envelopes in its various positions. I first observe that, if one gives the triangle $PR_1 R_2$ an infinitely small movement, or if one deplaces it infinitely little from its actual position, the result will be that the sides of this triangle will turn, or will tend to turn around the points of contact Q_1, Q_2, G which belong to them respectively. But, abstracting for a moment from the curve wich in general the side $R_1 R_2$ will envelope, it is evident that this side will also tend to roll [here Poncelet refers to the argument we have summarized in Part I of the proof of the Lemma] around a conic which has a double contact with the circle C; therefore the point of contact of that conic and the side $R_1 R_2$ is also the one of that same side with the unknown curve; and consequently, if one traces the straight lines $R_1 Q_1$, $R_2 Q_2$ and if one joins the point H of their crossing with the vertex P by the line PH, its prolongation will meet that of $R_1 R_2$ in the point of contact G which concerns us here [here Poncelet refers to the construction we have summarized in Part I of the proof of the Lemma]." (Our translation)

Now from part II of the proof we know that the circle through G, I_3 and I_4 touches $R_1 R_2$, and hence D, also in G. Therefore, in all of its points G, the curve D touches the circle through G and I_3 and I_4. Hence either D envelops the family of circles through I_3 and I_4, or D coincides with one of these circles. The former cannot occur because the family does not have an envelope, hence D is a circle through I_3 and I_4. Thus the Lemma is proved in the case of circles with real intersections; but by the principle of continuity it may be considered to apply in general. This completes Poncelet's proof of the Lemma; the main theorem now follows by projective generalization, and the general theorem is proved in the way explained in 4.1.

4.4 After having given the proof of the Lemma Poncelet remarked (Section 533) that for each P on C there are in general four different possible positions for the triangle $PR_1 R_2$, hence four different chords $R_1 R_2$ that envelop circles from the pencil. However, the four circles thus arising coincide pairwise (cf. Sections 9.5–7). Poncelet proved this by a symmetry argument: Consider (Figure 4.2) the triangle $P' R_1' R_2'$ symmetric to $PR_1 R_2$ with respect to the horizontal axis in the figure. When P is moved along C to P', $R_1 R_2$ does not transform into $R_1' R_2'$. Hence these two chords belong to two different starting positions of the describing triangle. Still $R_1' R_2'$ touches the circle D, which means that that in its motion it envelops D. Hence the envelopes arising in these two different cases coincide (as, by a similar argument, do the other two).

Poncelet also separately discussed (Section 532) the exceptional cases in which the envelope D degenerates. He referred to earlier Sections (76, 80, 370) about pencils of conics or of circles, in which he had shown that such pencils have three degenerate elements, namely (if we denote the four common points by $I_1, \ldots, I_4$) the line pairs $I_1 I_2 \cup I_3 I_4$, $I_1 I_3 \cup I_2 I_4$, and $I_1 I_4 \cup I_2 I_3$. He concluded this from a generalization (by the principle of continuity) of the case of pencils of real circles, and sometimes he spoke of the intersections K, L, M of the line pairs as infinitely small limit circles of the pencil, rather than of the line pairs themselves. However, it is clear that he was aware of both interpretations, and that he considered lines through K, L, M as tangents to the degenerate conics. [Cf. the Specialization Argument 9.9, where the same concept of tangents to degenerate conics is used.]

As to the cases of degeneration in the general theorem Poncelet noted that the envelope D may degenerate, and that this happens in particular in the case of a pencil of real circles with two real intersections, when for some P on C, $R_1 R_2$ coincides with $I_3 I_4$. Other cases of degeneration, he said, are equivalent to this case by the law of continuity. [In Section 9.4 we will find as condition for the degeneration of D that for some k, l, $k \neq l$, the tangents at I_k to D_1 and at I_l to D_2 intersect on C; it is easily seen that that condition is the same as the one mentioned by Poncelet.]

Poncelet also remarked that if D_1 coincides with D_2 (which is the case in the proof of the closure theorem) one of the two enveloped conics D coincides with C (cf. 9.11.2).

Poncelet had noted that the conics D_i' enveloped by the diagonals in the proof of the general theorem may degenerate. Hence in order to apply the main theorem in the induction proof, he should have considered the cases where D_1, D_2 or both degenerate. He did not do so, but it seems most likely that he was aware of that possibility, and that he would invoke the principle of continuity to proclaim the main theorem valid also in these exceptional cases (as indeed it is, cf. Remark 9.11).

We may assume that Poncelet was aware of all the special cases that may occur in connection with the main theorem. That he did not discuss all of them explicitly is in keeping with his general attitude to such special cases. That attitude is well illustrated by the following quotation, taken from his discussion of degenerate conics in pencils. He claimed that it is rather pointless to study these cases in detail, because

> "it is evident that these concepts and these properties will remain valid in an analogous way, and with modifications that will always be indicated by the law of continuity and by an attentive study of what happens when the general figure passes to the particular one" ([1822] Section 98; our translation)

Poncelet's first, analytical proof of the closure theorem

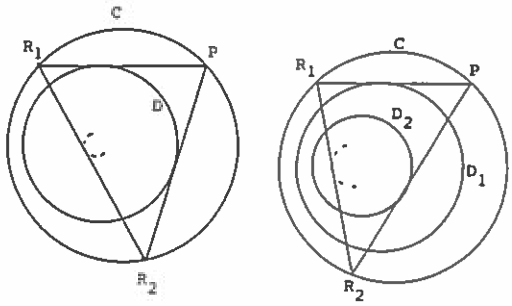
4.5 As we have indicated in Section 3.1, Poncelet found and proved both the "main theorem" and the closure theorem during the period of his Russian captivity 1813–1814. His first proofs of these theorems were completely different from the proofs which he chose to publish in the *Traité* and which we have summarized above. In Saratov he proved the theorems by straightforward if immensely laborious computation. These proofs can be found in [Poncelet 1862] vol. 1, pp. 308–372 (it is the sixth of the "Cahiers" from the Saratov period which Poncelet published in 1862). We shall restrict ourselves here to indicating the main lines in the calculations.

Poncelet first studied the case of circles (see Figure 4.4). He considered two circles C and D with equations

$$x^2+y^2=r^2 \quad \text{and} \quad (x-a)^2+y^2=R^2.$$

He chose $P=(x(t), y(t))$ on C for some parameter t, and calculated the equations of PR_1, PR_2 and $R_1 R_2$, where R_i are on C and PR_i are tangent to D. He then showed, by differentiating the equation of $R_1 R_2$ with respect to the parameter t, that the $R_1 R_2$ envelop a circle belonging to the pencil defined by C and D.

Figure 4.4



Poncelet then proceeded to the case of three circles C, D_1, D_2 from one pencil (Figure 4.4), calculating directly the equation of $R_1 R_2$; where again P, R_1 and R_2 are on C and PR_i are tangent to D_i. At this point the calculations became exceedingly involved, indeed the formulas as printed in [1862] required sheets folding out to the width of four quarto pages (e.g. p. 336), and even so they could be given only by help of many abbreviations for separate terms and factors. The formulas were too involved to admit calculating the envelope of $R_1 R_2$ by differentiation, but Poncelet was able to show that there is a point on the axis whose distance to all $R_1 R_2$ is the same, thus proving that the envelope is again a circle, belonging, as he also showed, to the same pencil.

After this analytical proof of the theorem in the case of circles and $n=3$, Poncelet generalized these results, and derived the closure theorem from them, in much the same way as he did later in the *Traité*; the case for general n was obtained by induction, the case for general conics by projection, the closure theorem by letting all D_i coincide. We quote how Poncelet formulated the closure theorem in 1813–14:

> "Geneally speaking it is impossible to inscribe in a given curve of second degree a polygon which is at the same time circumscribed to another curve of that degree, and when the particular disposition of these curves will be such that the simultaneous inscription and circumscription is possible for one single poly-

gon tried at will, then that guarantees that there will be an infinity ((of polygons)) with the same property with respect to the given conics." ([1862], *1*, p. 355; our translation)

The "Cahier" concludes with miscellaneous remarks, among other things about the cases of "trivial" closure that can occur if one of the vertices (sides) is a common point (tangent) of the two conics (cf. Sections 7.6–9).

These analytic proofs are important because they testify how fully Poncelet accepted analytical methods in the beginning of his career as a geometer, and hence how strong was the change of programme when he decided to write the *Traité* in a fully synthetic way. It should be noted that in his [1889] (p. 9) Loria expressed doubts that Poncelet knew the theorem before 1817 (which is curious because Loria had access to Poncelet's [1862]). As virtually all other surveys of the history of the theorem rely on Loria's study, it seems to have been unknown that Poncelet actually found and proved the closure theorem already in 1812–1814 during his Saratov period.

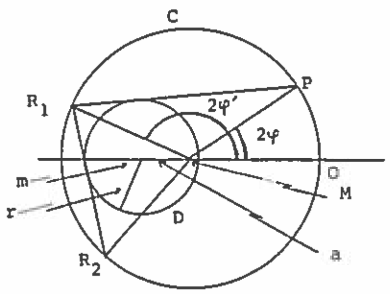
5. Jacobi

5.1 Some years after Poncelet's *Traité*, Jacobi published an article [1828] in which he proved Poncelet's closure theorem by means of elliptic functions. Jacobi found his new method while working out analytical formulas for chords and tangents in the case of a Poncelet traverse between circles. In doing so he recognized certain relations that also occur in the theory of elliptic functions. Pursuing this insight he found that it almost directly yielded full analytic proofs of Poncelet's theorems (both the "general theorem" and the closure theorem) and of the relations between R, r and a for circles admitting interscribed polygons.

In this section we shall explain how Jacobi came to see the link with elliptic function theory and we shall summarize his proof.

5.2 Following Jacobi's argument, we consider (see Figure 5.1) two circles C and D, D lying within C, with radii R and r ($R>r$), centres M and m, and distance of the centres a ($r+a<R$). The line through the centres of C and D intersects C in O.

Figure 5.1



Let PR_1 be a chord of C which touches D. Let $\angle OCP=2\phi$ and $\angle OCR_1=2\phi'$ (angles measured counterclockwise). By applying elementary trigonometry we derive

$$R\cos(\phi'-\phi)+a\cos(\phi'+\phi)=r.$$

Repeating the tangent construction from R_1, with $\angle OCR_2=2\phi''$, we find similarly

$$R\cos(\phi''-\phi')+a\cos(\phi''+\phi')=r.$$

These formulas are transformed into

$$\begin{aligned}&(R+a)\cos\phi'\cos\phi+(R-a)\sin\phi'\sin\phi=r\\&(R+a)\cos\phi''\cos\phi'+(R-a)\sin\phi''\sin\phi=r,\end{aligned}\tag{5.1}$$

respectively. Subtraction, applying the relation

$$(\cos x-\cos y)/(\sin y-\sin x)=\operatorname{tg}[(x+y)/2],$$

and remodelling yields

$$\operatorname{tg}[(\phi''+\phi)/2]=[(R-a)/(R+a)]\operatorname{tg}\phi'.\tag{5.2}$$

This formula reminded Jacobi of a formula occurring in the theory of elliptic functions. Let, for a certain k, $0<k<1$

$$u=F(\phi)=\int_0^{\phi}\frac{dt}{\sqrt{1-k^2\sin^2 t}}$$

be the elliptic integral of the first kind. Then define the inverse function am by

$$\mathrm{am}\,u=\phi.$$

Consider values ϕ, ϕ' ϕ'' χ, u and c such that

$$\phi=\mathrm{am}\,u$$

$$\chi=\mathrm{am}\,c$$

$$\phi'=\mathrm{am}(u+c)$$

$$\phi''=\mathrm{am}(u+2c),$$

then we have

$$\operatorname{tg}(\phi+\phi'')/2)=\Delta\,\mathrm{am}\,c\cdot\operatorname{tg}\phi',\tag{5.3}$$

with

$$\Delta\,\mathrm{am}\,c=[1-k^2\sin^2\chi]^{1/2}=[1-k^2\sin^2(\mathrm{am}\,c)]^{1/2}.$$

(Formula 5.3, as well as 5.4 below are easily derived from the usual addition formulas for the functions am, cn (=cosam) and sn (=sinam); for which see e.g. [Byrd and Friedman, 1954] p. 23.) The correspondence of Formulas 5.2 and 5.3 suggested to Jacobi the possibility to adjust k in such a way that the successive values of ϕ, ϕ', ϕ'', ..., corresponding to the vertices P, R_1, R_2 ... of a Poncelet traverse are related as

$$\phi = \operatorname{am} u$$

$$\phi' = \operatorname{am}(u+c)$$

$$\phi'' = \operatorname{am}(u+2c)$$

$$\phi^{(m)} = \operatorname{am}(u+mc),$$

for some u and c. Note that Jacobi found the analytic expression for the function $\phi \to \phi'$ in a Poncelet traverse by deriving the functional equation (5.2) and noticing that the function $\operatorname{am} u \to \operatorname{am}(u+c)$ is a solution of that equation. Apparently Jacobi considered it obvious that the solution is unique.

5.3 We shall now summarize the results which Jacobi gained by working out this correspondence. In doing so we shall keep to his ideas, but not to the order and detail of his presentation.

Jacobi did not proceed from Formulas 5.2 and 5.3 but rather from another relation from the theory of am and F. If

$$\phi = \operatorname{am} u$$

$$\phi' = \operatorname{am}(u+c)$$

$$\chi = \operatorname{am} c,$$

then

$$\cos\phi\cos\phi' + \sin\phi\sin\phi'[1-k^2\sin^2\chi]^{1/2} = \cos\chi.$$

Comparing this equation with Formula 5.1 written in the form

$$\cos\phi\cos\phi' + \sin\phi\sin\phi'\cdot[(R-a)/(R+a)] = r/(R+a),$$

we adjust χ and k such that

$$\cos\chi = r/(R+a)$$

and

$$1-k^2\sin^2\chi = (R-a)^2/(R+a)^2,$$

which yields

$$k^2 = 4aR/[(R+a)^2 - r^2].$$

For one outer circle, that is, for fixed R, different inner circles will in general yield different values of k^2. But Jacobi made the crucial observation that if C, D_1 and D_2 belong to one pencil of circles, then the pairs C, D_1 and C, D_2 yield the same k^2, and conversely, every k^2 determines a pencil through C. The proof is easy: Let

$$C: \quad x^2+y^2=R^2$$
$$D_1: \quad (x+a_1)^2+y^2=r_1^2$$
$$D_2: \quad (x+a_2)^2+y^2=r_2^2,$$

then C and D_i intersect in two (in our case imaginary) points whose x-coordinates are

$$x_i=[r_i^2-R^2-a_i^2]/2a_i.$$

For circles from one pencil the x_i are constant, say $x_i=x$, so that

$$k^2=2R/(R-x),$$

so k^2 is constant.

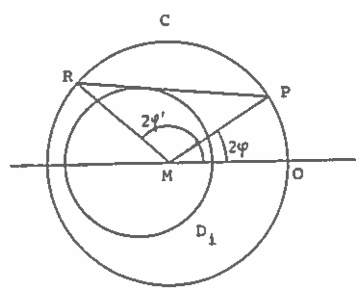
[Jacobi proved this in a different way, taking over, from Steiner, the definition of a pencil of circles as a set of circles C_i such that there is a straight line L in the plane, from all of whose points the tangents to the C_i are equal. He remarked that this amounts to what Poncelet expressed by saying that the circles have common real or imaginary chords (cf. Section 4.1).]

From now on we will consider a circle C and a number of circles D_i lying within C and belonging, together with C to one pencil. Let W_i be the mapping geometrically defined by

$$W_i(\phi)=\phi'$$

if P and R on C, with $\angle OMP=2\phi$, $\angle OMR=2\phi'$ are such that PR is tangent to D_i (see Figure 5.2). In other words, W_i describes the mapping $P\rightarrow R$ from C to C along chords tangent to D_i. As to the choice of the tangent from P, we follow Jacobi who tacitly supposed that we turn counterclockwise, that is, that $\phi'-\phi$ has minimal positive value.

Figure 5.2



We may now infer that it is possible to chose real numbers c_i such that, if $\phi=\mathrm{am}\,u$

then

$$W_i(\phi) = \mathrm{am}(u + c_i).$$

To be explicit, we choose c_i such that

$$\mathrm{am}c_i = \chi_i$$

and

$$\cos\chi_i = r_i/(R + a_i).$$

Note that c_i is dependent on the position of circle D_i, but not on the value of ϕ. Note further that for all D_i the modulus k^2 is the same, so that indeed the same function am occurs for each D_i.

Jacobi noted a converse of the preceding result, namely: if W is defined by

$$W(\phi) = \mathrm{am}(u + c); \qquad \phi = \mathrm{am}u$$

for some c, then the chords PR of C with $\angle OCP = 2\phi$ and $\angle OCR = 2W(\phi)$ envelop a circle D from the pencil corresponding to C and the modulus k of am. The proof is obvious; taking

$$\mathrm{am}c = \chi$$

there is precisely one circle D (with radius r) from the pencil such that

$$\cos\chi = r/(R + a).$$

The tangency-mapping W_D connected with D is:

$$W_D(\phi) = \mathrm{am}(u + c)$$

so that W_D coincides with W. The corresponding chords therefore envelop the circle D.

5.4 Jacobi interpreted the converse result mentioned at the end of the P receding section also in a different way, namely as providing a geometrical construction for the addition law of the elliptic function am. His argument may be summarized as follows.

For the elliptic function am, with modulus k^2 we want to find a geometrical construction by which, given

$$\phi = \mathrm{am}u \quad \text{and} \quad \chi = \mathrm{am}v,$$

we can find

$$\psi = \mathrm{am}(u + v).$$

[In fact Jacobi showed how to construct $\psi_n = \mathrm{am}(u + nv)$; he called this the "problem of the multiplication of elliptic functions" ([1828] p. 286).]

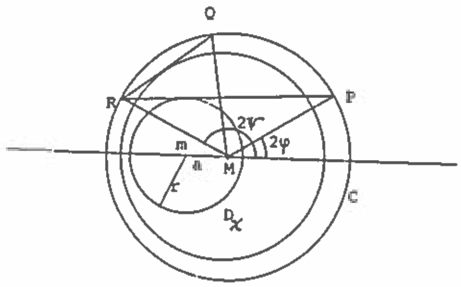


Figure 5.3

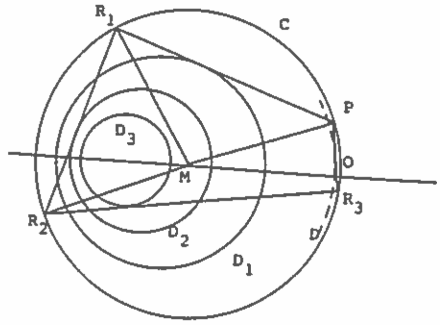


Figure 5.4

We take (see Figure 5.3) a circle C with radius R and centre M on the axis MO. k^2 defines a pencil $P_{C,k}$ of circles ($C \in P_{C,k}$), namely those with radii r, centre m on the axis, $mM = a$, and

$$k^2 = 4aR/[(R+a)^2 - r^2].$$

The given angles ϕ and χ define points P and Q and C by

$$\angle OMP = 2\phi \quad \text{and} \quad \angle OMQ = 2\chi.$$

The angle χ also determines a circle D_χ from $P_{C,k}$ by

$$\cos\chi = r/(R+a), \qquad 1 - k^2 \sin^2\chi = (R-a)^2/(R+a)^2.$$

Now draw from P the tangent to D_χ (counterclockwise) and determine its second intersection R with C. Then

$$\operatorname{am}(u+v) = \psi = (\angle OMR)/2,$$

so we have found ψ. We note that the same point R will be found if we start from Q and draw a tangent to D_ϕ; in other words $\operatorname{am}(u+v) = \operatorname{am}(v+u)$.

Jacobi's construction is interesting because in this argument he comes nearest to a structure isomorphic to the elliptic curve that plays a crucial role in the modern proof of the closure theorem. We will return to this matter in Section 11.5.

5.5 We now turn to Jacobi's proof of

Theorem 5.5.1. *General Theorem. Let (see Figure 5.4) C, $D_1, \ldots, D_n$ be circles from one pencil; all the D_i lying within C. For all P on C form a traverse $P, R_1, \ldots, R_n$, with R_i on C and $PR_1, R_1R_2, \ldots, R_{n-1}R_n$ tangent to $D_1, \ldots, D_n$ respectively. Close the traverses by drawing the chords R_nP. Then these chords R_nP will envelop a circle D from the pencil.*

Proof (Jacobi). Let $\angle OMP = 2\phi$ and $\phi = \operatorname{am} u$; call $\angle OMR_n = 2\phi^{(n)}$. Then $\phi^{(n)}$ is determined by

$$\begin{aligned}\phi(n) &= W_n \circ W_{n-1} \circ \ldots \circ W_1(\phi)\\ &= \mathrm{am}(u+c_1+\ldots+c_n)\\ &= \mathrm{am}(u+c)\end{aligned}$$

for $c=c_1+\ldots+c_n$. The chords PR_n now correspond to a mapping W defined by

$$\phi=\mathrm{am}\,u, \qquad W(\phi)=\phi^{(n)}=\mathrm{am}(u+c),$$

so that these chords envelop a circle D from the pencil.

From the main theorem we could prove Poncelet's closure theorem in the spirit of Poncelet's own approach as follows:

Theorem 5.5.2. *Closure theorem. Let (see Figure 5.5) C and D be two circles, D within C. Let, for a certain P_0 on C the interscribed polygon $P_0, R_1, R_2, \ldots, R_{n-1}\,P_0$ close. Then from all P on C the interscribed n-gon will close.*

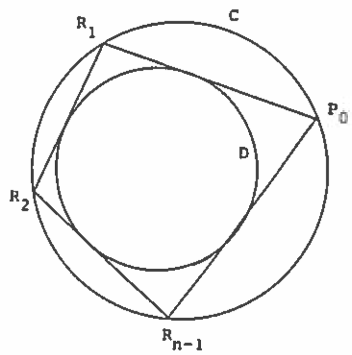


Figure 5.5

Proof. Consider the case of the main theorem with all C_i coinciding in D. Then, for variable P on C, the chords PR_{n-1} will envelop a circle D_0 from the pencil defined by C and D. D_0 and D have P_0R_{n-1} as common tangent. In the case of a pencil defined by two non intersecting circles, circles with a common tangent coincide. Hence D_0 and D coincide; therefore, starting from any P, the last side of an almost interscribed n-gon between C and D will also be tangent to D.

Jacobi, however, did not take this approach but argued analytically, highlighting the insights gained from his use of elliptic functions. The essence of his argument is as follows:

Proof of the closure theorem (Jacobi). Let W be the chord-tangent mapping associated to D. Consider a traverse of n chords starting at P and ending at R_{n-1}. Let $\angle OMP=2\phi$ and $\angle OMR_{n-1}=2\psi$. We have then, for some u and c,

$$\psi=\mathrm{am}(u+nc).$$

Now let for P_0 with angle ϕ_0 the traverse close to become an interscribed n-gon. That means that $P_0=R_{0,n-1}$; or, for certain natural number s:

$$\begin{aligned}\operatorname{am}(u+nc) &= \phi_0 + s\pi \\ &= \operatorname{am} u + s\pi \\ &= \operatorname{am}(u+2sK),\end{aligned}$$

where

$$K = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2 \sin^2 t}}.$$

Hence

$$u + nc = u + 2sK,$$

or

$$c = 2sK/n.$$

This relation is necessary and sufficient for closure of an interscribed line series after n steps and the relation is independent of the starting point P, because ϕ does not occur in it. Indeed if the traverse closes for some P_0 we have for any P with angle ϕ:

$$\begin{aligned}\angle OMK_n/2 &= \operatorname{am}(u+nc) \\ &= \operatorname{am}(u+2sK) \\ &= \operatorname{am} u + s\pi \\ &= \phi + s\pi,\end{aligned}$$

so that $R_{n-1} = P$. This concludes Jacobi's proof of the closure theorem.

6. Miscellaneous remarks

6.1 In this section we collect a number of short remarks on the studies of Poncelet and Jacobi and on the further history of the closure theorem.

Jacobi derived the general theorem and the closure theorem for circles lying within each other, not for circles that intersect. He was aware that by Poncelet's projection theorem (3.1, 3.4.2) the results can be generalized to conics but he did not work out this generalization as far as possible. He quoted Poncelet's theorem as: any two conics that have not more than two intersections can be projected on two circles ([1828], p. 281). This shows that Jacobi was hesitant to take over the results which Poncelet gained by applying his principle of continuity. After having derived his results for circles, Jacobi remarked that they can be generalized by projection to the case of ellipses lying within each other. This means that Jacobi here had real conics in mind and did not consider an extension to complex projective space. At the end of the article he suggested, as a topic for further research, to derive the analytic formulae directly in the case of conics. In that case the integral corresponding to $F(\phi)$ would be more complicated but reducible to $F(\phi)$. Jacobi announced that he might return to this matter, but as far as we have been able to judge from his published work he did not do so. Perhaps Jacobi did not try to work out his approach to this fullest generality because he considered the problem of interscribed polygons as belonging to elementary geometry.

6.3 Jacobi used his analytical approach also to find the relations

$$f_n(a, r, R) = 0$$

necessary and sufficient for two circles to have interscribed n-gons. Let C and D be two such circles; we have then, as derived in 5.4.

$$c = 2sK/n,$$

where

$$K = \int_0^\pi \frac{dt}{\sqrt{1-k^2 \sin^2 t}},$$

$$k^2 = 4aR/(R^2 + a^2 - r^2)$$

and

$$\mathrm{am}c = \chi = \arccos[r/(R+a)].$$

These equations implicitly determine the required relation

$$f_n(a, r, R).$$

6.4 Before determining f_n in this general way Jacobi surveyed these results by earlier mathematicians on this question. It is here that he brought the prehistory of the Poncelet closure theorem in connection with the closure theorem itself (cf. Section 2.4). He attributed f_3 to Euler, mentioned the work of Fuss on symmetrical interscribed polygons and stated that, because of the Poncelet closure theorem, Fuss's results are indeed general. He also discussed Steiner's results, compared them with those of Fuss and deplored the fact that Steiner did not give his derivation of the formulas ([Jacobi 1828] pp. 279–283).

6.5 Many mathematicians have taken up Jacobi's results and developed them further. The history of these later studies is very complicated; for surveys of it see [Loria 1889], [Dingeldey 1903] pp. 46–52 and [Kötter 1901] pp. 139–153. We will restrict ourselves here to noting only the main themes within these later developments.

Generalizing the results on $f_n(a, r, R)$, mathematicians tried to find the conditions which two conics have to satisfy in order to admit an interscribed n-gon. Here invariant theory, which was being developed during the second half of the nineteenth century, proved useful (admitting an interscribed n-gon is an invariant property). Cayley reached a significant result here, cf. [Griffiths, Harris 1978].

The fact that transcendental relations, namely elliptic functions, entered the proofs of a theorem which concerned purely algebraic geometrical objects, namely conics, led mathematicians to attempt purely algebraic proofs for the closure theorem, or to study the deeper reasons for the appearance of transcendental relations in this case.

Finally, mathematicians pursued further generalisations, as for instance to circles on spheres, or to surfaces in space.

6.6 When he prepared the edition [1862] of his geometrical studies before 1822, Poncelet added a note ([1862] vol. 1, pp. 480–488) on Jacobi's proof of the closure theorem and on other studies about it. The note is written with bitterness about the lack of recognition for his own approach to geometry. Although praising Jacobi's proof, Poncelet maintained that the algebraic and transcendental analytic methods for deriving geometrical results were too complicated and sometimes misleading, and that they could often be avoided, or at least be shortened, by utilizing more fully the results of his *Traité*.

We learn from the note that Jacobi visited Poncelet in 1829 and explained his methods to him. Poncelet also wrote that Steiner had told him that he (Steiner) had induced and encouraged Jacobi to use the theory of elliptic functions to prove the theorems from Poncelet's *Traité* about interscribed polygons. This provides an interesting link between the work of Steiner, Poncelet and Jacobi.

7. A modern proof of Poncelet's closure theorem

In this section we recall a modern treatment of the proof of the closure theorem of Poncelet; in the next sections we shall analyse steps in the proofs by Poncelet and by Jacobi with the help of modern notation and methods.

We use algebraic geometry over a field k, which we assume *algebraically closed* and of *characteristic zero* (the reader may assume $k=\mathbb{C}$, the field of complex numbers). By the word "conic" we understand a smooth plane curve of degree 2 (hence the union of two lines will not be considered as a conic). When $\mathbb{P}=\mathbb{P}^2$, the projective plane, we denote

$$\mathbb{P}^* := \{L \mid L \text{ is a line in } \mathbb{P}^2\}$$

and, of course $\mathbb{P}^* \cong \mathbb{P}^2$. When $X \subset \mathbb{P}^2$ is an algebraic curve, we denote by

$$X^* \subset \mathbb{P}^*$$

the dual curve, i.e., let $Y \subset X$ be the (open) subset of smooth points of X, then

$$Y^* := \{L \mid L = T_{X,P} \quad \text{for some } P \in Y\}$$

(where $T_{X,P}$ is the tangent line at P to X), and X^* is the closure of Y^* (we assume we have a topology, the reader may take $k=\mathbb{C}$, and take the complex topology or we could take the Zariski-topology on $\mathbb{P}^2$, cf. [Hartshorne, 1977], p. 10). Sometimes the dual of an algebraic curve is not so easy to describe (cf. [Walker, 1950), IV. 5; cf. [Griffiths and Harris, 1978a], 2.4; cf. [Brieskorn and Knörrer, 1981], pp. 321 – 333), however,

$$\text{when } D \subset \mathbb{P}^2 = \mathbb{P} \text{ is a conic then } D^* \subset \mathbb{P}^* \text{ is a conic}$$

(this is not difficult), and for any algebraic curve $X \subset \mathbb{P}^*$ as above, the map

$$X \supset Y \to X^*$$
$$P \mapsto L = T_{X,P}$$

is bijective outside a finite set (and hence birational because $\operatorname{char}(k) = 0$). Further note that,

$$\text{if } C \subset \mathbb{P}^2 \text{ is a conic,} \quad \text{then } C \cong \mathbb{P}^1$$

(cf. [Hartshorne, 1977], p. 30, Exercise 4.4a; cf. [Walker, 1950], III.5.1; this really is very easy).

Suppose given $C, D \subset \mathbb{P}^2$ two different (smooth) conics. We define (see Figure 7.1):

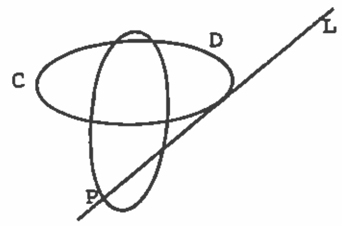


Figure 7.1

$$E(C, D) := \{(P, L) \mid P \in C, L \in D^*, P \in L\} \subset C \times D^*.$$

Lemma 7.1. If $\# (C \cap D) = 4$ then E is a smooth curve, and it is an elliptic curve.

Remark 7.2. The number of intersection points (counted with multiplicity) of C and D equals $2.2 = 4$ (by Bezout), and the condition $\# (C \cap D) = 4$ is equivalent with: if $P \in C \cap D$ then $T_{C,P} \neq T_{D,P}$ (i.e. "C and D are nowhere tangent").

Proof of the lemma. Consider the "projection"

$$E(C, D) \to C$$
$$(P, L) \mapsto P,$$

this map is 2:1 iff $P \notin C \cap D$ (if $P \notin D$, then there exist 2 tangent lines to D through P, here we use $\operatorname{char}(k) \neq 2$). Clearly $E \subset C \times D^*$ is an algebraic curve, and we claim: if $(P, L) \in E$, and $P \notin C \cap D$ then E locally at (P, L) is isomorphic with C locally at P. (If we consider these curves as Riemann surfaces, with the complex topology we mean locally in that sense; if we use the Zariski-topology we mean that the completion of the local ring $O_{E,(P,L)}$ is naturally isomorphic with the completion of the local ring $O_{C,P}$; the proof of this is not difficult, and it is left to the reader.) Conclusion:

$$E-\{(P,L)\mid P\in C\cap D\}$$

is smooth.

In the same way we conclude that E locally at (P, L) is isomorphic with D^* locally at L iff $L\notin C^*\cap D^*$, and we conclude analogously:

$$E-\{(P,L)\mid L\in C^*\cap D^*\}$$

is smooth. Note that we assumed

$$\{(P,L)\mid P\in C\cap D \quad \text{and} \quad L\in C^*\cap D^*\}=\emptyset,$$

thus: E is smooth. Finally we show that E is elliptic, i.e.

$$g(E)=1$$

(here g is the genus). We give two proofs; consider the projection $E\rightarrow C$, we have seen that this map is 2:1 except in the 4 points $\{(P,L)\mid P\in C\cap D,\ L=T_{D,P}\}$; thus by the Zeuthen-Hurwitz formula (cf. [Mumford, 1976], p. 142, 7.20; cf. [Hartshorne, 1977], p. 301, 2.4) we have

$$2g(E)-2=2\cdot(2g(C)-2)+4,$$

and because $g(C)=0$, we obtain $g(E)=1$. We can also argue as follows

$$E\subset C\times D^*\cong\mathbb{P}^1\times\mathbb{P}^1$$

and $\{P\}\times D^*$ and $C\times\{L\}$ have intersection multiplicity 2 with E; thus E is called a curve of type (2, 2) on $\mathbb{P}^1\times\mathbb{P}^1$, the canonical divisor K on $\mathbb{P}^1\times\mathbb{P}^1$ is of type $(-2,-2)$ ([Hartshorne, 1977], p. 188, Exc. 8.4*e*), and the adjunction formula on $\mathbb{P}^1\times\mathbb{P}^1$ gives (cf. [(Hartshorne, 1977], p. 361, 1.5):

$$2g(E)-2=F\cdot(E+K)=(2,2)\cdot((2.2)+(-2,-2))=0,$$

hence $g(E)=1$. (We gave this second argument because we shall later see curves of type (2, 2) on $C\times C\cong\mathbb{P}^1\times\mathbb{P}^1$.)

Notation 7.3. Suppose given C, $D\subset\mathbb{P}^2$, two different conics. We write γ: $E\rightarrow E$ for the morphism defined by (see Figure 7.2)

$$\gamma(P,L)=(P',L)$$

i.e. γ is the involution on E which commutes with the covering $E\rightarrow C$. Analogously

$$\delta: E\rightarrow E$$

$$\delta(P,L)=(P,L').$$

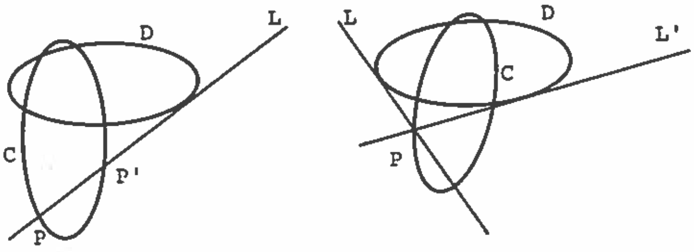


Figure 7.2

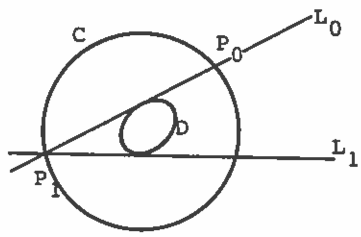


Figure 7.3

Note 7.4: The fixed points of γ are all points

$$(P, L) \text{ with } P = P', \text{ i.e. } L \in C^* \cap D^*,$$

and the fixed points of δ are all points

$$(P, L) \text{ with } L = L', \text{ i.e. } P \in C \cap D.$$

Note 7.5: Suppose we start with (see Figure 7.3)

$$(P_0, L_0) \in E = E(C, D),$$

then take $P_1 = P'_0$ (the second point of intersection of L_0 with C), then take $L_1 = L'_0$ (the second tangent line through P_1 to D), then

$$(P_1, L_1) = \delta\gamma(P_0, L_0),$$

this follows immediately from the definition of γ and of δ.

Construction and definition 7.6. Suppose given $C, D \subset \mathbb{P}^2$, two different conics. If

$$P_0 \in C, \quad L_0 \in D^* \quad \text{with} \quad P_0 \in L_0,$$

and $n \in \mathbb{Z} \geq 3$ we construct (the "Poncelet traverse", cf. Section 1.1):

$$P_0, P_1, \ldots, P_n \in C \quad \text{and} \quad L_0, L_1, \ldots, L_{n-1} \in D^*$$

with

$$L_i = P_i P_{i+1}, \quad 0 \leq i \leq n$$

(i.e. given P_i and L_i, we take for P_{i+1} the second point of intersection, $L_i \cap C = \{P_i, P_{i+1}\}$; it may happen that L_i touches C, and then $P_i = P_{i+1}$ and given P_{i+1} and L_i we take for L_{i+1} the second tangent line to D, thus $(P_{i+1})^* \cap D^* = \{L_i, L_{i+1}\}$, etc.).

We say that this construction *closes after n steps* (or that the polygon given by $P_0, \ldots, P_{n-1}$ is *interscribed* between C and D) if

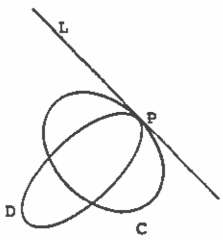
$$P_n = P_0$$

(i.e. if $P_{n-1}\, P_0 = L_{n-1}$). We say that this construction *closes trivially* (cf. [Van der Waerden, 1973], p. 139) if

$$P_j = P_{n-j}, \quad \text{for all } j \quad \text{with} \quad 0 \leq j \leq n.$$

We say that the closing is *degenerate* if $P \in C \cap D$, $L \in C^* \cap D^*$, $P \in L$: in that case $P_0 = P$ gives $P_0 = P_i$ for all i (see Figure 7.4).

Figure 7.4



Remark 7.7: The cases of trivial closing are the following:

if $n = 2k+1$ is *odd* and $L_k \in C^* \cap D^*$,

or

if $n = 2m$ is *even* and $P_m \in C \cap D$.

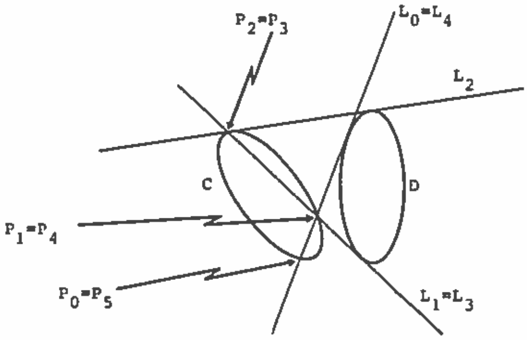


Figure 7.5

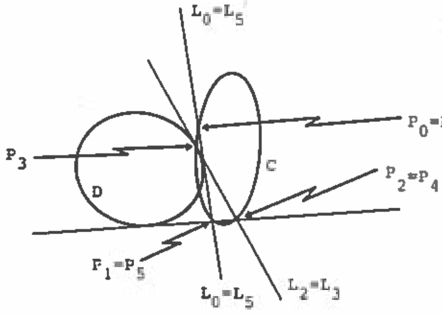
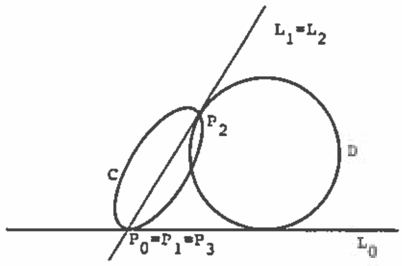


Figure 7.6

Example 7.8: It may happen that the n-gon "folds up" without the closing being trivial. Suppose (see Figure 7.7) $L_0 \in C^* \cap D^*$, $P_0 \notin C$, etc., and $P_2 \in C \cap D$; if $P_2 \neq P_0$ the triangle $\{P_0, P_1, P_2\}$ gives a non-trivially closing construction.

Figure 7.7



Remark 7.9: If the closing is degenerate then it is trivial.

Lemma 7.10. Suppose given $P_0 \in C$, $P_0 \notin D$, and $L_0 \in D^*$, $P_0 \in L_0$. The n-construction closes non-trivially if and only if

$$(\delta\gamma)^n(P_0, L_0) = (P_0, L_0).$$

Proof. Suppose the construction closes trivially, then

$$\gamma(\delta\gamma)^{n-1}(P_0, L_0) = (P_0, L_0);$$

by $P_0 \notin D$ we conclude $\delta(P_0, L_0) \neq (P_0, L_0)$, thus

$$(\delta\gamma)^n(P_0, L_0) \neq (P_0, L_0).$$

Conversely suppose the n-gon closes non-trivially; if we would have $(\delta\gamma)^n(P_0, L_0) \neq (P_0, L_0)$ then

$$\gamma(\delta\gamma)^{n-1}(P_0, L_0) = (P_0, L_0).$$

Suppose $n = 2k+1$ is odd, then

$$(\delta\gamma)^k\, \gamma(\delta\gamma)^{2k}(P_0, L_0) = (\delta\gamma)^k(P_0, L_0)$$

gives

$$(P_{k+1}, L_k) = (P_k, L_k),$$

i.e. the construction closes trivially. Suppose $n = 2m$ is even, then

$$(\delta\gamma)^m\, \gamma(\delta\gamma)^{2m-1}(P_0, L_0) = (\delta\gamma)^m(P_0, L_0)$$

gives

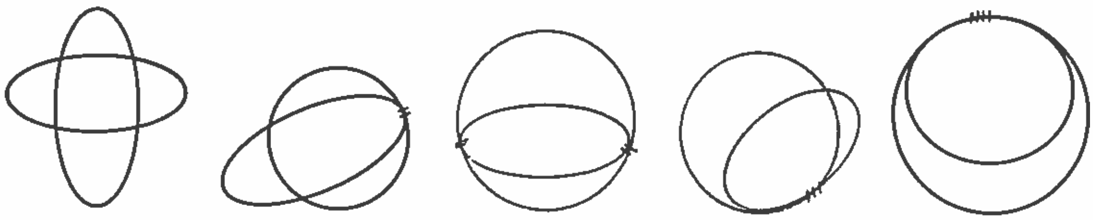
$$(P_m, L_{m-1}) = (P_m, L_m),$$

i.e. the construction closes trivially. We obtain a contradiction in both cases, thus if the n-gon closes non-trivially then $(\delta\gamma)^n(P_0, L_0)$.

Theorem 7.11. *Poncelet's closure theorem (cf. Section 1.1). Suppose $C, D \subset \mathbb{P}^2$ are two different (smooth) conics. Suppose for some $n \in \mathbb{Z}_{\geq 3}$ there exists a non-trivial interscribed n-gon between C and D (i.e. there exists $P_0 \in C$, $L_0 \in D^*$, $P_0 \in L_0$ such that the construction closes non-trivially after n steps). Then for any $S_0 \in C$, and for any $M_0 \in D^*$ with $S_0 \in M_0$ the construction closes after n steps.*

Different positions 7.12. We know C intersects D in 4 points with multiplicities counted. Thus we can have the following cases (and it is not difficult to show that all cases occur):

1) $4 = 1+1+1+1$, $\#(C \cap D) = 4$, 2) $4 = 2+1+1$,
3) $4 = 2+2$, 4) $4 = 3+1$, 5) $4 = 4$.



1 2 3 4 5

Figure 7.8

Remark 7.13. These situations are self-dual, i.e. if C and D in $\mathbb{P}^2$ have a certain intersection behavior, then C^*, $D^* \subset (\mathbb{P}^2)^*$ have the same (proof left to the reader).

Proof of Poncelet's closure theorem, case (1) (We give arguments which can be found in [Griffiths, 1976], I.d, and in [Griffiths and Harris, 1977]; the relation of this modern proof to those by Poncelet and by Jacobi will be discussed in Section 11.) We have seen that in case (1), the curve E is an elliptic curve (smooth, of genus $=1$). It is known that such a curve is a group variety, and that the group structure is unique once $0 \in E$ is chosen ([Walker, 1950], VI.9; [Hartshorne, 1977], IV.4). Note that

$$\alpha = \delta\gamma : E \xrightarrow{\sim} E,$$

and note that $\delta\gamma$ has no fixed points, because

$$\delta\gamma(P, L) = (P, L)$$

would imply that $L \in C^* \cap D^*$ and $P \in C \cap D$, which cannot happen in case (1). We write $\alpha = \delta\gamma$.

Claim. *Choose any $0 \in E$, there exist $t \in E$ such that*

$$\alpha x = x + t$$

for all $x \in E$ (and $+$ with respect to $0 \in E$).

First proof of the claim: Let $\alpha(0) =: t$; the map $x \mapsto \alpha(x) - t$ fixes zero, thus

$$\beta : (E, 0) \to (E, 0), \qquad \beta x = \alpha(x) - t$$

is a homomorphism (cf. [Hartshorne, 1977], p. 322, Lemma 4.9). Suppose $-\beta + id$ would be non-zero; then

$$(-\beta + id) : E \to E$$

is surjective (because $\dim E = 1$, and E being a complete curve, $(-\beta + id)\,E$ is closed in E; cf. e.g. [Hartshorne, 1977], p. 103, Th. 4.9); thus we can choose $x_0 \in E$ with

$$(-\beta + id)(x_0) = t.$$

Then

$$t=(-\beta+id)\,x_0=-\beta(x_0)+x_0=-\alpha(x_0)+t+x_0,$$

thus

$$x_0=\alpha(x_0),$$

a contradiction with the fact that α has no fixed points.

Second proof of the claim: We choose $0\in E$, let $c\in E$ be a fixed point of γ, thus $c=(P,L)$ with $P\in C\cap D$, and let $d\in E$ be a fixed point of δ, thus $d=(Q,M)\in E$ with $M\in C^*\cap D^*$. Let $+$ be the group structure on E with 0 as zero element. By [Hartshorne, 1977], p. 322, (4.9) we deduce

$$\gamma x=-x+2c, \qquad \delta y=-y+2d;$$

thus

$$\delta\gamma x=\delta(-x+2c)=x+t, \qquad t:=2d-2c,$$

and the claim is proved.

Now suppose the n-gon $P_0, \ldots P_n=P_0$ closes non-trivially. Then we conclude

$$(\delta\gamma)^n(P_0,L_0)=(P_0,L_0)$$

(if $P_0\notin D$, use Lemma 7.2; if $P_0\in D$, then closing implies $\gamma(\delta\gamma)^{n-1}(P_0,\ L_0)=(P_0,\ L_0)$ and $P_0\in D$ gives $\delta(P_0,L_0)=(P_0,L_0)$). For any $x\in E$ we have

$$\delta\gamma x=x+t$$

thus taking $x=(P_0,L_0)$, from

$$(\delta\gamma)^n(P_0,L_0)=(P_0,L_0)$$

we deduce

$$x+nt=x, \qquad \text{thus } nt=0.$$

Take $S_0\in C$, choose $M_0\in D^*$, with $S_0\in L_0$, and let $y=(S_0,M_0)\in E$; then

$$(S_n,M_n)=(\delta\gamma)^n(S_0,M_0)=Y+nt=y=(S_0,M_0),$$

thus the n-gon $\{S_0,\ldots,S_n=S_0\}$ closes. Q.E.D.

The closure theorem in cases of tangency 7.14. We sketch the proof of the closure theorem in the other cases. We denote by $E=E(C,D)$ the incidence curve as before, and we write E^0 for the smooth part of it. Up to choice of a zero-point on E^0, this has a unique structure of a group variety in each of the cases and the curve E and the group structure on E^0 are as follows.

Type		E	Group
1	$4=1+1+1+1$	Elliptic	
2	$4=2+1+1$		G_m
3	$4=2+2$		$G_m\times(\mathbb{Z}/2)$
4	$4=3+1$		G_a
5	$4=4$		$G_a\times(\mathbb{Z}/2)$

Figure 7.9

where G_m = multiplicative linear group, i.e. $G_m=\mathbb{P}^1-\{0,\infty\}$ as a variety, and the group structure is the multiplication of the coordinates on $\mathbb{P}^1$, and $G_a=\mathbb{P}^1-\{\infty\}$, and addition. The proof of these facts is straightforward: E is smooth above points where C and D are not tangent; in points where C and D have a common tangent we shall determine the singularity of E by a direct computation.

We choose coordinates such that D is given by the equation

$$Y=X^2,$$

$$\text{if } Q=(t,t^2), \quad \text{then } L=T_{D,Q}:2tX=Y+t^2;$$

we take C in the form

$$\alpha Y=X^2+\beta XY+\gamma Y^2, \quad \alpha\neq 0,$$

$$P=(\alpha s/(1+\beta s+\gamma s^2),\alpha s^2/(1+\beta s+\gamma s^2)) \text{ on } C;$$

note that $(0,0)\in C\cap D$, the curves are tangent there, and their intersection number at that point is given by the length of the ring

$$k[X,Y]/(Y-X^2,-\alpha Y+X^2+\beta XY+\gamma Y^2)\cong k[X]/(X^2-\alpha X^2+\beta X^3+\gamma X^4)$$

localized at $(0,0)$, hence:

type (2) or (3) if $1-\alpha\neq 0$,

type (4) if $1=\alpha$ and $\beta\neq 0$,

type (5) if $1=\alpha$, $\beta=0$, and $\gamma\neq 0$.

Note that $D\cong D^*$, the coordinates t and s are coordinates on $D\times C\cong D^*\times C$, and in these coordinates E is given (by $P\in L$ hence) by:

$$2t\alpha s/(1+\beta s+\gamma s^2)=\alpha s^2/(1+\beta s+\gamma s)+t^2$$

thus by

$$(\alpha s^2-2\alpha st+t^2)+\beta st^2+\gamma s^2t^2=0.$$

The discriminant of the quadratic term is $4\alpha^2-4\alpha=4\alpha(\alpha-1)$; because $\alpha\neq 0$ (C is irreducible), and because $4\neq 0$ ($\mathrm{char}(k)\neq 2$) we see:

$1-\alpha\neq 0$ iff the singularity is a *node*.

If $1=\alpha$, the equation is

$$(s-t)^2+\beta st^2+\gamma s^2t^2=0$$

which is an (*ordinary*) *cusp* iff $\beta\neq 0$. If $1=\alpha$ and $\beta=0$ we obtain

$$(s-t)^2+\gamma s^2t^2=(s-t+\sigma\tau\sqrt{-\gamma})(s-t-st\sqrt{-\gamma})=0$$

and we obtain a *tacnode*. In case (3) the curve E has two nodes, and $E\to C$ does not ramify outside the nodes; one concludes easily E to be reducible.

In each of the cases consider γ and δ, and $\alpha=\delta\gamma$ as above. Note that both γ and δ interchange the two components of E in the cases (3) and (5). Hence

$\alpha\in\mathrm{End}(\mathbb{P}^1-\{0,\infty\})$ in (2), (3),

$\alpha\in\mathrm{End}(\mathbb{P}^1-\{\infty\})$ in (4), (5).

This map is birational (bijective) and it has no fixed points. In the last case

$$\alpha u=au+b;$$

because it has no fixed points, we see $a=1$, thus

$$\alpha u=u+b;$$

in the first case

$$\alpha v=av \quad \text{or} \quad \alpha v=a/v;$$

if $\alpha v=a/v$, then $\sqrt{a}$ is a fixed point, hence

$$\alpha v=av.$$

If $\{P_0,\ldots,P_n=P_0\}$ is a closing n-gon which closes non-trivially, then this closing is non-degenerate (i.e. $P_0P_1=L_0\notin C^*$). Thus (P_0,L_0) is *not* a singular point of E and the proof of the closure theorem is as before.

Remark 7.15. Note that in cases (2) and (3) the Poncelet n-gon closes if and only if $a^n=1$. Note that in cases (4) and (5) (and supposing $\mathrm{char}(k)=0$), the Poncelet n-gon does not close: if $\alpha^n u=u$, then $nb=0$, thus $b=0$.

Remark 7.16. If we work over a field K with $\mathrm{char}(K)=p>2$, the whole proof works, and we find no closing p-gons in cases (2) and (3), and we find that in every situation the cases (4) and (5) have closing n-gons for $n=p$.

Note that the case (3) can be proven much easier than we did: by a projective transformation we map the two intersection points to the isotropic points $(1:\pm i:0)$, with $i=\sqrt{-1}$, the conics C and D become concentric circles and rotation-symmetry immediately proves the closure theorem in this case.

Remark 7.17. The morphism $E=E(C, D)\to C$ ramifies above $C\cap D$, and one can decide the structure of E from this; e.g. consider C fixed, let D vary such that for general position $\#(C\cap D)=4$ (and hence E is elliptic) and for special position $\#(C\cap D)=3$ (type (2), the limiting curve has a node as is well known) etc. In this way we see what happens with closing n-gons under this "specialization": we know exactly what happens with torsion points under specialization, and we see that any closing n-gon in situation (2) or (3) be obtained by specialization of a closing n-gon in situation (1): we sketch a proof for situation (2) as a limiting position. Choose $C\subset\mathbb{P}^2$ and P_1, P_2, $P_3\in C$ fixed. Let P_4 be a variable point on C, and let $\mathscr{E}\to C$ be the family of curves such that $\mathscr{E}_\alpha$ is the curve which is determined by $P_4=\alpha\in C$ such that

$$\mathscr{E}_\alpha\to C$$

is a 2:1 covering branched in P_1, P_2, P_3 and $P_4=\alpha$. Choose a family $D\subset\mathbb{P}^2\times C$ given such that D_α passes through P_1, P_2, P_3 and $P_4=\alpha$ (take generic α, choose for that value D_α and specialize). Let V be the variety whose points correspond to triples (α, β, d) with $\alpha\in C$, $\beta\in\mathbb{A}^1$ and $D_{2,\beta}$ is the conic given by $\beta C+D_\alpha$, and d (depending on α and β) is the point

$$d=(Q, M)\in E_\alpha=E(C, D_{\alpha,\beta})$$

with M a common tangent of C and $D_{\alpha,\beta}$ (cf. Figure 7.10). We can always choose d so that it corresponds to a common tangent not passing through $P_4=\alpha$. Note that $V\to C\times\mathbb{A}^1$ at the generic point is 4:1 (corresponding to the 4 choices of d). Note that C and $D_{\alpha,\beta}$ give a Poncelet construction which closes after n steps if $nt=0$ (here t can be constructed from C, $D_{\alpha,\beta}$ and d).

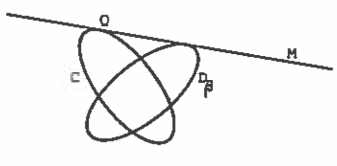


Figure 7.10

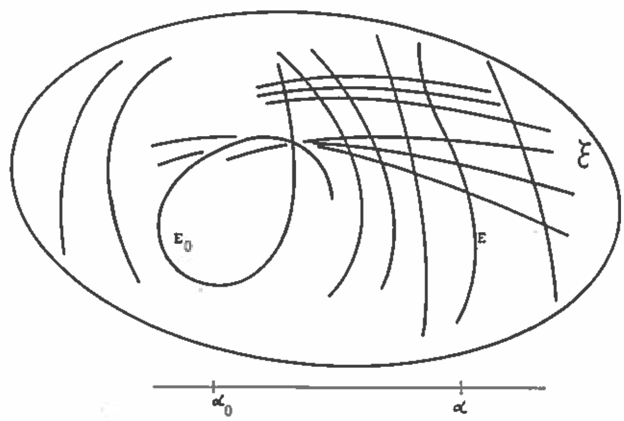


Figure 7.11

Now consider the case of tangency $P_4 = P_1 = \alpha_0$. Then the curves $E_{\alpha_0,\beta} = E(C, D_{\alpha_0,\beta})$ have a node, whereas for general α the curves $E_{\alpha,\beta}$ are elliptic. It is well known (theory of "vanishing cycles" as developed by Lefschetz, cf. [Lefschetz, 1924]; e.g. also cf. [Igusa, 1958]) that *all* n-torsion points of the nodal curve E_0,

$$(E_0)^0[n] \cong \mathbb{Z}/n,$$

are specializations of n-torsion points of the generic fibre

$$E_\alpha[n] \cong (\mathbb{Z}/n) \times (\mathbb{Z}/n);$$

thus, if α_0, β_0 is a "closing situation" we can choose (α, β, d) specializing to (α_0, β_0, d_0), and the corresponding pair C, $D_{\alpha,\beta}$ is a "closing situation". We indicate by a figure the theory of "vanishing cycles": cf. Figure 7.11.

Remark 7.18. We have seen $E \subset C \times D^* \cong \mathbb{P}^2 \times \mathbb{P}^1$; one can embed this surface as a quadric in $\mathbb{P}^3$ (see Figure 7.10),

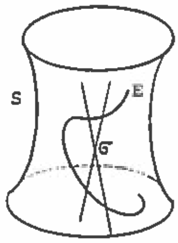
$$\mathbb{P}^1 \times \mathbb{P}^1 \cong S \subset \mathbb{P}^3.$$

S is the set of zeros of any non-degenerate quadratic polynomial (homogeneous in 4 variables). Choose $\sigma \in S$, and project $\mathbb{P}^3$ on a $\mathbb{P}^2$ with centre σ (map not defined at σ). This projection restricted to S we denote by $\pi = \pi_\sigma$,

$$\pi: S - \{\sigma\} \to \mathbb{P}^2.$$

If $\sigma \in E$, then $\pi(E - \{\sigma\})$ is a cubic, smooth curve in $\mathbb{P}^2$. If $\sigma \in E$, then $\pi(E)$ is a quadric curve in $\mathbb{P}^2$ with 2 singular points (the two lines on S through σ collapse to points under π_σ, and each of these lines has intersection multiplicity 2 with E. This is the situation described in [Griffiths, 1976], pp. 345/346.

Figure 7.12



Remark 7.9. Another proof of Poncelet's closure theorem can be given with the help of the "correspondence principle" of Chasles, cf. [Van der Waerden, 1973], pp. 138/139.

8. Poncelet's infinitesimal argument

Poncelet's proof of the closure theorem contains a beautiful argument about infinitesimal motions and rotations (cf. Section 4.3, part III of the proof of the lemma, in particular Poncelet's own text quoted there). In the present section we shall formalize his argument, show that it applies not only to conics but to algebraic curves in general, and deduce some consequences.

Modern algebraic geometry supplies some tools for this. Consider the ring $k[\varepsilon]/(\varepsilon^2)$ (the ring of "dual numbers over k", k is a field, and $(a+b\varepsilon)(c+d\varepsilon)=ac+(ad+bc)\varepsilon$). We can formulate geometric "infinitesimal" arguments by using this ring as the ring of constants (instead of working over a field).

Lemma 8.1. *Let C and D be algebraic curves in $\mathbb{P}^2$, let $P\in C$, $Q\in D$ be smooth points and let L be a line which is tangent to D at Q, with $P\in L$. Suppose: $P\neq Q$, and: Q is not a flex on the curve D. Let $P_\varepsilon\in C(k[\varepsilon])$ be an infinitesimal deformation of $P\in C(k)$ (i.e. P_ε has coordinates $x^{(i)}\in k[\varepsilon]$ which satisfy the equation for C, and under the ring homomorphism $k[\varepsilon]\to k$, $\varepsilon\mapsto 0$, we have $P_\varepsilon\mapsto P$). Let L_ε be the tangent to D deforming L. Let M_ε be the line joining P_ε and Q (it "pivots" around Q). Then*

$$L_\varepsilon = M_\varepsilon$$

(e.g. as elements of $(\mathbb{P}^2)^*(k[\varepsilon])$).

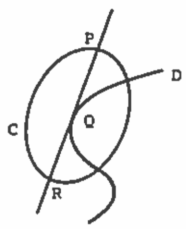


Figure 8.1

Proof. We choose coordinates in $\mathbb{A}^2\subset\mathbb{P}^2$ such that $P=(0,1)$, $Q=(0,0)$, L is given by $X=0$; let $P_\varepsilon=(u\varepsilon, 1+v\varepsilon)$, with $u, v\in k$. An equation for M_ε is

$$\begin{vmatrix} u\varepsilon & 0 & X \\ 1+v\varepsilon & 0 & Y \\ 1 & 1 & 1 \end{vmatrix} = (1+v\varepsilon)X - u\varepsilon Y = 0.$$

Note that $1-v\varepsilon$ is a unit in the ring $k[\varepsilon]$, thus

$$(1-v\varepsilon)[(1+v\varepsilon)X-u\varepsilon Y] = X - u\varepsilon Y$$

defines the same line over $k[\varepsilon]$. A polynomial defining D reads

$$G\equiv \alpha X+\beta Y^2 \pmod{X^2, XY}$$

with $\alpha\neq 0$ (because D is smooth at $(0,0)=Q$), and with $\beta\neq 0$ (because Q is not a flex of D, so $X=0$ intersects with multiplicity 2). If L_ε is tangent to D at Q_ε, then

$$Q_\varepsilon=(0, w\varepsilon)$$

(because $Q_\varepsilon \in D$ its X-coordinate equals zero), and L_ε is given by a polynomial

$$(\partial G/\partial X)_{Q_\varepsilon} X + (\partial G/\partial y)_{Q_\varepsilon} Y + \text{const},$$

thus

$$H = \alpha X + 2\beta w \varepsilon Y;$$

by $P_\varepsilon \in L_\varepsilon$ we get

$$\alpha u \varepsilon + 2\beta w \varepsilon (1 + v\varepsilon) = 0$$

because of $\beta \neq 0$ we can solve w knowing u,

$$w = -\alpha u/(2\beta),$$

and

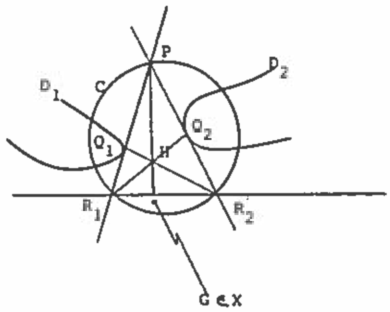
$$(1/\alpha) H = X + (1/\alpha) 2\beta w \varepsilon Y = X - \varepsilon u Y,$$

thus $L_\varepsilon = M_\varepsilon$. Q.E.D.

Remark 8.2. If $\beta = 0$ (i.e. D has a flex at Q), and if C is not tangent to $X = 0$ at P, then a *non-trivial* deformation P_ε on C, over the ring $k[\varepsilon]/(\varepsilon^2)$ cannot be obtained by any deformation of Q and $T_{D,Q}$. In fact, if $\beta = 0$ and $Q_\varepsilon(0, w\varepsilon)$, the corresponding deformation L_ε of $T_{D,Q}$ is given by $H = \alpha X$; by $P_\varepsilon \in L_\varepsilon$ we conclude $\alpha u \varepsilon = 0$ in $k[\varepsilon]/(\varepsilon^2)$; by $\alpha \neq 0$ this leads to $u = 0$; because C is not tangent to $X = 0$ at P this leads to $P_\varepsilon = (u\varepsilon, 1 + v\varepsilon) = (0, 1)$ (we need higher order deformations of Q and $T_{D,Q}$ in order to have a non-trivial deformation of P). Note that in the rest of the paper Lemma 8.1 will be applied in case D is a conic (hence smooth and without inflection points).

Construction 8.3. Suppose given (cf. Figure 8.2) a (smooth) conic $C \subset \mathbb{P}^2$ and plane algebraic curves $D_1, D_2 \subset \mathbb{P}^2$ (they may be equal). Suppose D_1 and D_2 do not have components which are a line. We define $\Gamma \subset (\mathbb{P}^2)^* \cong \mathbb{P}^2$ and $X \subset \mathbb{P}^2$ (depending on C, D_1 and D_2) in the following way. For any $P \in C$, and a pair of lines L_1, L_2 so that $P \in L_i$, and L_i tangent to D_i, $i = 1,2$ we write $L_i \cap C = \{P, R_i\}$. Let X be the set "enveloped by the chords $R_1 R_2$". By this we mean the following. For all P, L_1, L_2 with the properties above, and so that $R_1 \neq R_2$ we define the line $R_1 R_2 \in (\mathbb{P}^2)^*$, and the union of all points thus defined gives a subset $\Gamma^0 \subset (\mathbb{P}^2)^*$. The closure of Γ^2 we denote by $\Gamma \subset (\mathbb{P}^2)^*$. We show that Γ is an algebraic curve: let $F^0 \subset D_1^* \times D_2^*$ be the set of pairs $(L_1, L_2) \in D_1^* \times D_2^*$ so that $L_1 \neq L_2$ and $L_1 \cap L_2 \subset C$, let F be the closure of F^0; it is easily seen that F is an algebraic curve. If $(L_1, L_2) \in F^0$, then the construction applies with $P := L_1 \cap L_2$, $P \in C$, we obtain $R_1 R_2 \in (\mathbb{P}^2)^*$ and this defines a morphism $F^0 \to (\mathbb{P}^2)^*$. The closure of the image is Γ, and it is easily seen that all components of Γ have dimension one, hence Γ is an algebraic curve. We define $X = \Gamma^*$, i.e. for any line contained in Γ we obtain a point in X, and all other components of X are algebraic curves. If $Q_i \in D_i$, with $Q_i \notin C$ so that $L_1 \cap L_2 = \{P\} \subset C$, and $L_i := PQ_i$ is tangent to D_i, $i = 1, 2$, with $L_1 \neq L_2$, we construct $H := Q_1 R_2 \cap Q_2 R_1$ and $G := PH \cap R_1 R_2$ (cf. Figure 8.2; because of the conditions the lines $Q_1 R_2$ and $Q_2 R_1$ are defined and are different, thus H is defined, and $P \neq H$ and $PH \neq R_1 R_2$, hence G is defined). Note that the conditions $Q_i \notin C$ give a dense open subset of F^0 (only finitely many points have to be omitted). We shall see cases where X actually has a component of dimension zero.

Figure 8.2



We now formulate the analogon of Poncelet's result on the construction of points on the envelope X (cf. Section 4.3, especially parts I and III of the proof; we do not use that D_1 and D_2 are conics).

Proposition 8.4. *Suppose C, D_1, D_2 as in the Construction 8.3, suppose the points P, R_1, R_2, Q_1, Q_2 are mutually different (see Figure 8.2), suppose Q_i is not a flex of D_i, $i=1,2$. Then the point G as constructed, i.e.*

$$Q_1 R_2 \cap Q_2 R_1 = H, \qquad PH \cap R_1 R_2 = G,$$

is a point of contact between X and $R_1 R_2$, and

$$G \in X \cap R_1 R_2.$$

Proof. Because $R_1 \neq R_2$, we can draw the line $R_1 R_2 = \gamma \in (\mathbb{P}^2)^*$, $\gamma \in \Gamma$. Suppose P_ε is a non-trivial deformation of $P \in C$, $\varepsilon^2 = 0$. We apply the previous lemma ($P \neq Q_i$, Q_i is not a flex of D_i, $i=1,2$), concluding that the tangent line deformation $L_{i,\varepsilon}$ is the same as $P_\varepsilon Q_i = M_{i,\varepsilon}$, the line through P_ε and Q_i defined over $k[e]$, $i=1,2$; we conclude:

$$L_{i,\varepsilon} = M_{i,\varepsilon};$$

because $Q_1 \neq R_1$ (or because $Q_2 \neq R_2$) we have $R_{1,\varepsilon} \neq R_1$ (or $R_{2,\varepsilon} \neq R_2$), thus

$$\gamma_\varepsilon := R_{1,\varepsilon} R_{2,\varepsilon} \neq \gamma,$$

so we obtain a non-trivial deformation γ_ε of $\gamma \in \Gamma \subset \mathbb{P}^2$. This $\gamma_\varepsilon \in \mathrm{Mor}(\mathrm{Spec}\, k[\varepsilon], \mathbb{P}^2)$, thus this non-trivial tangent vector γ_ε to Γ at γ is supported by a line N tangent at γ to Γ. The situation $\gamma \in N \subset (\mathbb{P}^2)^*$ corresponds dually with $\gamma \leftrightarrow R_1 R_2$, $N \leftrightarrow G \in R_1 R_2$. For fixed P on C, the γ and γ_ε in the case that $P_\varepsilon R_{i,\varepsilon}$ remain tangent to D_i, are the same as the γ and γ_ε that arise if $P_\varepsilon R_{i,\varepsilon}$ pivot around Q_i (by 8.1). Hence the point G of contact is the same in both cases. In Lemma

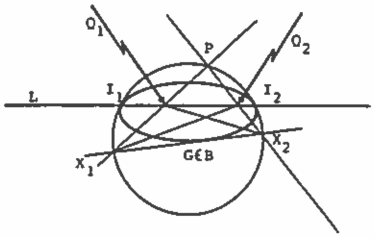
8.5 we show that in the latter (pivotting) case the point G is constructed as indicated in the proposition. It follows that also in the former case that construction applies.

Lemma 8.5. *Let C be a conic, Q_1 and Q_2 points not on C, let $L = Q_1 Q_2$ intersect C in points I_1, I_2, assume $Q_1 \neq Q_2$ and $I_1 \neq I_2$. For any point $P \in C$, $P \notin L$ we construct $X_1, X_2 \in C$ as indicated: the line $PQ_j = 1, 2$, intersects C in P and X_j.*

(8.5 i) All lines $X_1 X_2$ (for $P \in C$ variable) pass through one points iff the pair $\{Q_1, Q_2\}$ is harmonic with respect to $\{I_1 I_2\}$; in this case all lines $X_1 X_2$ pass through the polar point M of L with respect to C.

(8.5 ii) If $\{Q_1, Q_2\}$ is not harmonic with respect to $\{I_1, I_2\}$, the lines $X_1 X_2$ envelop a curve B, this is a smooth conic, it touches C in I_1 and I_2, and it touches $X_1 X_2$ in the point $G \in B$ as indicated in Figure 8.3: $H = X_1 Q_2 \cap X_2 Q_1$ and $G = X_1 X_2 \cap PH$.

Figure 8.3



Several proofs can be given, for example a straightforward computation is easy (parametrize C, compute the coordinates of X_1 and X_2, and eliminate the variable from the equation giving $X_1 X_2$, etc.). However, we prefer to give a proof more closely related to Poncelet's original idea; in (8.6) we indicate one detail in which we have to alter Poncelet's argument (and also cf. 4.3).

Proof. We assume the groundfield k is that of the complex numbers, $k = \mathbb{C}$. We choose coordinates in the projective plane containing C so that

$$I_1, I_2 = (1 : \pm i : 0), \qquad i = \sqrt{-1},$$

the *isotropic points*. The triple $\{C, Q_1, Q_2\}$ is uniquely determined by a point

$$(\alpha, \beta, \gamma, Q_1, Q_2) \in \mathbb{C}^3 \times L \times L,$$

where

$$(X + \alpha)^2 + (Y + \beta)^2 = \gamma \tag{1}$$

is the equation for C. Let $T \subset \mathbb{C}^3 \times L \times L$ be the subset where this equation is irreducible, and where $Q_1, Q_2 \notin C$. The points in T are exactly all those situations we study in the Lemma.

For any $t \in T$ we obtain

$$C_t \to \Gamma_t \in \mathbb{P}^*$$

by the construction, thus for each $t \in T$ the curve Γ_t is irreducible (being the image of C_t); and for each $t \in T$ we obtain $C_t \to B_t \subset \mathbb{P}$.

First step. Consider $S \subset T(\mathbb{R})$, where S is the set of points with real coordinates, and such that $\gamma > 0$. We claim that for all points in S the corresponding triple $\{C, Q_1, Q_2\}$ has the properties as concluded in the lemma. In fact, the pair $\{Q_1, Q_2\}$ is harmonic with the isotropic pair if and only if for every $P \in C(\mathbb{R})$, the lines PQ_1 and PQ_2 are perpendicular. This is the case if and only if for all $P \in C(\mathbb{R})$ the line $X_1 X_2$ contains the centre M of the circle C (cf. Figure 8.4). In all cases the angle between PQW_1 and PQ_2 is constant, and the point G given by the construction is the middle of $X_1 X_2$ (cf. Figure 8.5), thus for $P \in C(\mathbb{R})$ this middle describes a circle with centre M, and this circle degenerates to the point M iff $M = G \in X_1 X_2$. We see: if $t \in S$, and $\{Q_1, Q_2\}$ harmonic with respect to the isotropic points, the map $C \to B$ maps all the real points of C to M, thus it maps all points of C to M, and $B = \{M\}$ (because $\gamma > 0$ implies $C(\mathbb{R})$ consists of infinitely many points). In the case $\{Q_1, Q_2\}$ not harmonic it maps $C(\mathbb{R})$ to the real points of a circle, hence B equals this circle. The condition "$G \in B$" is a closed condition, it holds for all $P \in C(\mathbb{R})$, hence it holds for all $P \in C$. This finishes the proof of the lemma for all points $t \in S$.

Figure 8.4

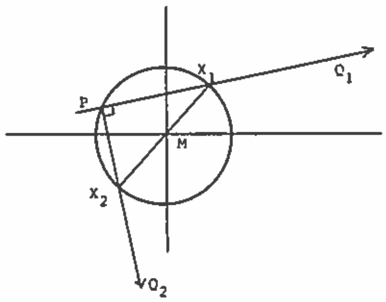
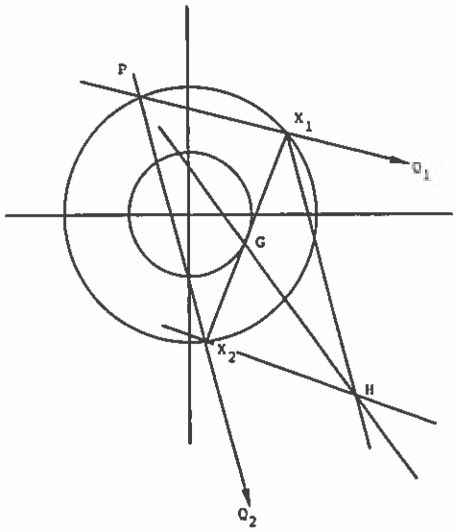


Figure 8.5



Second step. Let $T' \subset T$ be the set of points where $\{Q_1, Q_2\}$ is harmonic with respect to $\{I_1, I_2\}$ and let $T'' \subset T$ be the set of points where Γ_t is a line; the sets T' and T'' are $\mathbb{R}$-Zariski-closed in T as is easily seen. By the first step we know $T'(\mathbb{R}) \cap S = T''(\mathbb{R}) \cap S$; note that these are Zariski-dense subsets of T' and T''; thus we conclude $T' = T''$; note that the tangents to C in I_1 and I_2 pass through M. Thus we have proved (8.5i).

For all $t \in T$, $t \notin T' = T''$ the curve Γ_t is not a line. For all $t \in S$ we have seen in the first step that Γ_t is a conic, thus in the family

$$\bigcup_{t \in S} \Gamma_t \to S$$

for the Zariski-dense set S the fibres are irreducible, hence all Γ_t, $t\in S$, are irreducible, and we conclude that all $B_t:=(\Gamma_t)^*$, $t\in S$, and $t\notin T'$, are conics. The properties "$G_t\in B_t$" and "B_t touches C in I_1 and I_2" are closed conditions, and they are valid (by the first step) for all $t\in S$, thus they are true for all $t\in S$. This finishes the proof of the lemma.

Remark 8.6. If C', Q'_1, Q'_2 are defined over the real numbers, and we change coordinates (over $\mathbb{C}$) so that I_1 and I_2 are transformed into the isotropic points, the resulting Q_1, Q_2 need not have the property that they have real coordinates. Thus the first step in the proof does not show the lemma for all C', Q'_1, Q'_2 defined over the real numbers (cf. also 4.3).

Remark 8.7. Consider the situation as in 8.4. The line $R_1 R_2$ may have other points of contact with X. Thus we write "a point of contact ...". But we should say: take the "incidence curve" consisting of triples $(P, L_1, L_2)\in C\times D_1^*\times D_2^*$; this point (P, L_1, L_2) in the situation of 8.4 maps naturally onto $G\in X$ by the construction.

Remark 8.8. In the proof of Lemma 8.5 we have used the Zariski-topology to derive a result which Poncelet reached by means of his principle of continuity. It seems worthwhile to pursue further study of this principle of Poncelet: how much of it can be put on solid foundation e.g. by using the theory of Riemann surfaces or by the methods of modern algebraic geometry (such as the theory of schemes and the notion of the Zariski-topology, cf. also the proof of the Main Theorem 9.4.I).

9. A proof for Poncelet's "Main Theorem"

Let $C, D_1, D_2\subset\mathbb{P}^2$ be three different (smooth) conics, and consider the curve $X\subset\mathbb{P}^2$ as described in Construction 8.3 and in Proposition 8.4. Poncelet and Jacobi described X in the special case that C, D_1 and D_2 belong to the same pencil; in this section we study that case. In the next section we determine X in case C, D_1 and D_2 are "in general position". Let us refer to these cases in the following way.

Situation I: The three conics are different, I_1, I_2, I_3 and $I_4\in\mathbb{P}^2$ are 4 different points and

$$C\cap D_1=\{I_1, \ldots, I_4\}=C\cap D_2$$

(the conics belong to the same pencil).

Situation II: The conics are in general position.

Remark 9.1. By "general position" we mean the following. Suppose the situation is described by parameters (a conic in $\mathbb{P}^2$ is given by a homogeneous equation which has 6 coefficients, thus it can be given by a point in $\mathbb{P}^5$, and (C, D_1, D_2)

corresponds to a point in $\mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^5$). We say the situation is in general position if there is a finite union U of proper closed subvarieties U_i of the parameter space (this U depends on the problem considered) and the situation in question corresponds to a point not contained in U. So, to be correct one should specify first the problem, then define U, and after that the phrase "general position" makes sense. In our case, we give the construction of the situations we want to exclude (i.e. the closed set U) in the next section (and in the following lemma we already see one of the conditions).

Lemma 9.2.II. *Suppose* $\#(C \cap D_1) = 4 = \#(C \cap D_2)$ *and*

$$C \cap D_1 \cap D_2 = \emptyset.$$

Then

$$F := \{(P, L_1, L_2) \mid P \in C \cap L_1 \cap L_2,\ L_i \in D_i^*\}$$

is a smooth curve of genus 5.

Proof. We have $E(C, D_i) =: E_i \to C$ as in Section 7 and

$$F = E_1 \Pi_c E_2 = \{(P_1, L_1), (P_2, L_2) \mid P_1 = P_2\}.$$

The curves E_1, E_2 are smooth of genus one, and

$$F \to E_1$$

$$(P, L_1, L_2) \mapsto (P, L_1)$$

is generically (2:1) which ramifies only if $P \in C \cap D_2$; we conclude F is smooth outside these points. The same holds for F and $P \notin C \cap D_1$. Thus F is smooth be because $C \cap D_1 \cap D_2 = \emptyset$. The map $F \to E_1$ ramifies in exactly the 8 points (P, L_1) with $P \in C \cap D_2$. By the Zeuthen-Hurwitz formula (cf. [Hartshorne, 1977], p. 301) we have

$$2g(F) = (2g(E) - 2) \cdot n + 8(n-1)$$

and $n = 2$, $g(E) = 1$ yields $g(F) = 5$. Q.E.D.

Lemma 9.2.I. *In situation I,*

$$F = F_1 \cup F_2,$$

where F_1 and F_2 are elliptic curves intersecting transversally in 4 points.

Proof. A point $(P, L_1, L_2) \in F$, $P \notin C \cap D_1 = C \cap D_2$ is smooth on F (proof as before). Note that

$$E_1 \to C \quad \text{and} \quad E_2 \to C$$

are smooth curves, covering 2:1 over C, and both maps are branched in

$$C \cap D_1 = \{I_1, \ldots, I_4\} = C \cap D_2.$$

$$\begin{tikzcd}
E_1 \arrow[rr, shift left, "\phi"] \arrow[rr, shift right, "\psi"'] \arrow[dr] & & E_2 \arrow[dl] \\
& C &
\end{tikzcd}$$

Thus there exist exactly 2 isomorphisms over C (the curves E_1 and E_2 are isomorphic, and $\psi = \delta_2\, \phi$) and

$$F = \{(x, \phi x) \mid x \in E_1\} \cup \{(y, \psi y) \mid y \in E_1\}.$$

A local computation, which we omit, shows that at the points

$$I_i' = (I_i, L_1, L_2) \in F$$

these two components intersect transversally. Q.E.D.

Note that in case II we obtain morphisms

$$f: F \to \Gamma \subset (\mathbb{P}^2)^*$$
$$g: F \to X \subset \mathbb{P}^2$$

(these maps are defined on a non-empty open set of F, thus they extend to the set of smooth points of F, cf. [Hartshorne, 1977], p. 43, Prop. 6.8, thus in the situation as in 9.1.II to the whole of F). In Situation I we obtain

$$f^0: F - \{I_1', \ldots, I_4'\} \to \Gamma \subset (\mathbb{P}^2)^*$$
$$g^0: F - \{I_1', \ldots, I_4'\} \to X \subset (\mathbb{P}^2)$$

(and we shall see that g^0 does not extend).

The curve $F = F(C, D_1, D_2)$ defined by

$$F := \{P, L_1, L_2) \mid P \in C \cap L_1 \cap L_2, L_i \in D_i^*\}$$

is reducible in Situation I, $F = F_1 \cup F_2$ (cf. 9.1.I), and the curves F_1 and F_2 are smooth; thus we obtain morphisms

$$F_i \to f(F_i) = \Gamma_i \subset (\mathbb{P}^2)^*,$$

these give the components of Γ.

In Situation I we distinguish three cases. First we fix notation. We write

$$I = C \cap D_1 = C \cap D_2$$

(consisting of four points in $\mathbb{P}^2$, no three on a line). We choose one of the points in I, say $I_1 \in I$. We choose one of the conics $\{D_1, D_2\}$, say $D_1 \in \{D_1, D_2\}$. We draw the tangent line L_1 to D_1 at I_1, write $L_1 \cap C = \{P, I_1\}$ (and note: $P \notin I$, $P \notin D_2$). We write L_2 and L'_2 for the two tangents through P to D_2, with

$$L_2 \cap D_2 = \{Q_2\}, \qquad L_2 \cap C = \{P, R_2\},$$

$$L'_2 \cap D_2 = \{Q'_2\}, \qquad L_2 \cap C = \{P, R'_2\},$$

note that $R_2 \neq R'_2$ and that (P, L_1, L_2) and (P, L_1, L'_2) are on different components of F. We distinguish the three cases:

(I.a) We have $Q_2 \neq R_2$ and $Q'_2 \neq R'_2$ (thus $Q_2 \notin I$ and $Q'_2 \in I$) (cf. Figure 9.1).
(I.b)We have $Q_2 = R_2$ and $Q'_2 \neq R'_2$, or $Q_2 \neq R_2$ and $Q'_2 = R'_2$ (cf. Figure 9.2)
(I.c) We have $Q_2 = R_2$ and $Q'_2 = R'_2$.

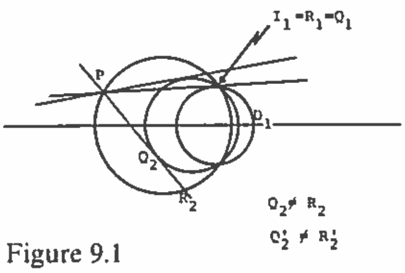


Figure 9.1

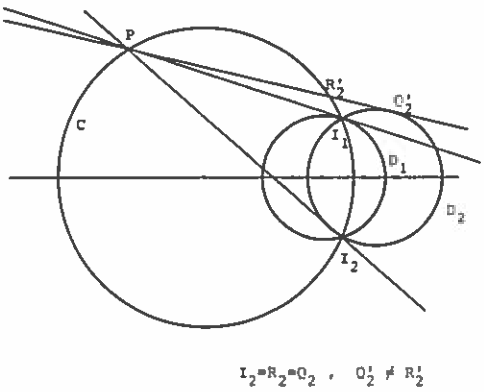


Figure 9.2

It seems that we treat the points in I asymmetrically, and also the same for D_1 and D_2. We show this is however not the case, and it turns out that these are all possibilities that can occur.

Remark 9.3. It is easy to see that cases (I.a) and (I.b) occur. The following example shows that (I.c) occurs. Let $I_1, I_3 = (0 : \pm 1 : 1)$, and $I_2, I_4 = (1 : \pm i : 0)$ with $i = \sqrt{-1}$, and

$$C: X^2 + Y^2 = 1,$$

$$D_1 : (X+1)^2 + Y^2 = 2,$$

$$D_2 : (X-1)^2 + Y^2 = 2.$$

We choose $P = (1:0:1)$, we see that the lines

$$X \pm Y = 1$$

touch D_1 in I_1 and $I3$, and the lines

$$Y = \pm i(X-1), \qquad i = \sqrt{-1}$$

touch D_2 in I_2 and I_4, and I_4, and $P \in C$ is on all these lines. Hence this is case (I.c).

Theorem (9.4.I) *("Main Theorem", cf. Sections 4.1 and 4.3). Let C, D_1 and D_2 be three different conics of the same pencil (i.e. Situation I). Then*

$$X = X_1 \cup X_2, \quad \text{with } X_1 \not\subset X_2 \quad \text{and} \quad X_2 \not\subset X_1,$$

where:

in Case (I.a): X_1 and X_2 are two conics in the same pencil;

in Case (I.b): X_1 is a conic in the same pencil and X_2 is a point;

in Case (I.c): the component X_1 and X_2 are two (different) points,
(and below we describe the exact positions of X_1 and X_2 in all cases).

Before giving the proof of the theorem, we need some preliminaries.

Lemma 9.5. *The symmetric group $S_4 = S(I)$ of permutations of the set I acts on $\mathbb{P}^2$*

$$S_4 = S(I) \hookrightarrow \mathrm{Aut}(\mathbb{P}^2),$$

and for the subgroup

$$V_4 \cong \{(1), (12)(34), (13)(24), (14)(23)\} \subset S_4$$

we have that any $\tau \in V_4$ maps any conic D containing I to itself. Any $\tau \in V_4$ maps

$$\tau: E_1 \to E_1, \qquad \tau: E_2 \to E_2$$

($E_i = E(C, D_i)$) and for $\tau \neq (1)$, $\tau \in V_4$ maps F_i and Γ_i to itself for $i = 1, 2$, and if $\tau \neq (1)$ it has no fixed points in F_i.

Proof. (Such a symmetry argument was already used by Poncelet in connection with pencils of conics, cf. Section 4.4 and the references to the *Traité* mentioned there.) Transform I_3 and I_4 to the isotropic points $(1:\pm\sqrt{-1}:0)$, and I_1 and I_2 to $(0:\pm 1:1)$; in the new coordinate system $\tau = (12)(34)$ is given by

$$\tau(x:y:z) = (x:-y:z)$$

and any D containing I is given by

$$\lambda(X^2 + Y^2 - Z^2) + \mu XZ = 0;$$

thus

$$\tau: D \to D.$$

It follows that τ: $E_i \to E_i$, $i = 1, 2$. *If* $(P, L_1) \in E_1$ with

$$\tau(P, L_1) = (P, L_1),$$

then

$$P=\tau P=(a:0:1)\in C,$$

and L is given by

$$\text{either } X=0 \quad \text{or} \quad X=aZ.$$

However, there is no smooth conic D_1 containing I which is tangent to either of these lines. Hence $\tau\in V_4$, $\tau\neq(1)$ has no fixed points on E_i. This implies that on E_1 (and on E_2) the map $\tau\in V_4$ is a translation by a point of order 2 (choose a base point in each of the curves, use the same arguments as in the proof of Theorem 7.11). Moreover we can choose $I_1^1\in E_1$, $I_1^2\in E_2$ as base points and then the points I_2^1, I_3^1, $I_4^1\in E_1$ are the points of order two on E_1, and the analogous statement for E_2 (with the notation: I_j^1 is the pair $I_j^1=(I_j, L)$ where L is the tangent to D' at I_j). Thus ϕ and $\psi: E_1\to E_2$ commute with $\tau\in V_4$, hence every $\tau\in V_4$ operates on the components F_1, F_2 of F:

$$\text{if } (P, L_1, L_2)\in F_i \quad \text{then } (\tau P, \tau L_1, \tau L_2)\in F_i.$$

The rest follows directly.

Lemma 9.6. *We have $C^*\not\subset\Gamma$, and for any line $p\subset\mathbb{P}^*$ with $p\not\subset\Gamma_i$ we have*

$$\#(f^{-1}(p\cap\Gamma_i))\leq 4 \quad \text{for } i=1,2.$$

Proof: If $P\in C$ is not on a common tangent of D_1 and D_2 we obtain (P, L_1, L_2) producing a chord R_1R_2 with $R_1\neq R_2$; thus the intersection is finite. If we choose $R\in C$ there are at most 8 points in F, at most four on each component F_i, such that R is the endpoint of a chord $K\in\Gamma$ (from R there are at most four tangents to D_1 or D_1, hence at most eight such points on F, and permuting L_1 and L'_1 or L_2 and L'_2 interchanges F_1 and F_2). As $C^*\not\subset\Gamma$ a general point $R\in C$ corresponds to a line $r\subset\mathbb{P}^*$ nowhere tangent to Γ, we see that a general line in $\mathbb{P}^*$ gives at most four points on each of the components of Γ.

Lemma 9.7. *Suppose $PI_1=L_1$ touches D_1 (in I_1), let $R_2\in C$, $R_2\neq P$, and suppose $PR_2=L_2$ touches D_2 in Q_2; we write $x_0:=(P, L_1, L_2)\in F_1$. Assume $Q_2\neq R_2$ (and hence $R_2\notin I$), see Figure 9.1. Then*

$$I_1R_2\cap X_1=\{I_1\}.$$

Proof: Note that F_1 is a smooth curve. Choose a variable point $x_t\in F_1$ with

$$\lim_{t\to 0} x_t=x_0$$

(either work over the complex numbers, consider F_1 as a Riemann surface, and take limits in the complex-analytical sense, or use the local ring of $x_0\in F_1$, this is a discrete valuation ring, and use specialization methods). For $x_t\in F_1$ we have

$x^t=(P^t, L_1^t, L_2^t)$, and for x^t in an open neighbourhood of x^0, but $x^t \neq x^0$, we are in the situation of (8.3) and (8.4), and we can construct H^t and $G^t \in X \cap R_1^t R_1^t$ in the way indicated by (8.3). Because F_1 is smooth

$$H_0 := \lim_{t \to 0} H^t \quad \text{and} \quad G_0 := \lim_{t \to 0} G^t$$

exist. Because $Q_2 \neq R_2$ we conclude

$$H_0 = I_1 = Q_2^0 R_1^0 \cap Q_1^0 R_2^0.$$

Because $H_0 = I_1 \neq P$, we conclude

$$G^0 = I_1 = PH^0 \cap R_2^0 R_1^0.$$

Thus

$$\lim_{t \to 0}(X \cap R_1^t R_2^t) = I_1 = X \cap R_1 R_2, \quad \text{Q.E.D.}$$

Lemma 9.8. *Each of the maps $f_i: F_i \to \Gamma_i$ is of degree at least two and $deg(\Gamma_i) \leq 2$ for $i = 1, 2$.*

Proof. Suppose there exists $i \in \{1, 2\}$ so that $4 \geq \deg(\Gamma_i) > 2$; then $f_i: F_i \to \Gamma_i$ is birational by (9.6), thus the geometric genus of Γ_i equals one (because $g(F_i) = 1$). If $\deg \Gamma_i = 3$ we see that $\tau \in V_4$, $\tau \neq (1)$ is a reflection in $(\mathbb{P}^2)^*$, hence it has fixed points on $\Gamma_i \cong F_i$, contradicting (9.5). If $\deg(\Gamma_i) = 4$, this curve has at most 2 singularities. Clearly V_4 cannot operate on $\Gamma_i \subset (\mathbb{P}^2)^*$ so that all $\tau \in V_4$, $\tau \neq (1)$ have no fixed points outside the singularities. Hence $\deg(\Gamma_i) \leq 2$. This implies $g(F_i) = 1 > 0 = g(\Gamma_i)$. Thus $F_i \to \Gamma_i$ is not birational. Q.E.D.

Proof of the main theorem (Case I.a). Suppose C, D_1 and D_2 chosen in such a way that for a choice of I_1 and D_1 we are in case (I.a). We see that $K = I_1 R_2$ and $K' = I_1 R_2'$ are chords such that K and K' are points on different components of Γ. Thus by (9.7) the two components of X contain I_1. Using the action of V_4 on the whole situation we see that both components of X contain all points of I. Hence $\deg \Gamma_1 = 2 = \deg \Gamma_2$, and X_1 and X_2 are conics through all points of I (and touching K and K' respectively at I_1, etc.). This finishes the proof in the Case (I.a).

Specialization argument 9.9. We parametrize all possible choices for Situation I, e.g. fix $I \subset \mathbb{P}^2$, a set of four points, no three on a line, in some standard position, take coordinates in the linear pencil of conics through the points in I, then

$$(C, D_1, D_2) \in T \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

where T is an open set. The subsets corresponding to the cases a, b, c have the property:

$$T = T^a \cup T^b \cup T^c, \qquad T^c \subset \overline{T^b} \subset \overline{T^a} = T.$$

Suppose we have some

$$(C^0, D_1^0, D_2^0) \in T,$$

we choose a smooth 1-parameter family

$$\{(C^t, D_1^t, D_2^t)\}_{t \in B},$$

parametrized by a smooth curve B so that this is the situation (C^0, D_1^0, D_2^0) for $t=0$, and for $t \in B$, $t \neq 0$ this point is in T^a (B is an open curve, and this is possible because T^a is dense in T). In this way we obtain a family

$$\{\Gamma^t\}_{t \in B},$$

and each fibre Γ^t is a plane algebraic curve. Moreover we have seen that for $t \neq 0$,

$$\Gamma^t = \Gamma_1^t \cup \Gamma_2^t, \qquad \text{2 conics.}$$

Further for $t \neq 0$

$$X^t = (\Gamma^t)^* = X_1^t \cup X_2^t,$$

two conics which contain the points in I, and which are tangent for chords as constructed before. We use the methods of "complete conics" (e.g. as explained in [Kleiman 1976] p. 470): if Γ^t degenerates to a (double) line Γ_i^0, then $X_i^t = (\Gamma_i^t)^*$ for $t \neq 0$ degenerates to a pair of lines and in that case $X_i^0 = (\Gamma_i^0)^*$ is a point. Note that in that case

$$\lim_{t \to 0}((\Gamma_i^0)^*) \neq (\lim_{t \to 0}(\Gamma_i^t))^*;$$

such a pair of lines and their intersection points both appear in the considerations by Poncelet, see Section 4.4.

Proof of the Main Theorem (Cases I.b and I.c). Suppose we are in one of the two special cases, (C^0, D_1^0, D_2^0), and choose a family degenerating to this case as above. Fix $I_1 \in I$ and D_1, and consider $K^t = I_1 R_2^t$ and $K'^t = I_1 R_2'^t$. We know that X_1^t and X_2^t are conics, containing the points of I, and touching K^t, respectively K'^t at I_1 for all $t \neq 0$. If this situation degenerates to (I.b) the conic touching IR_2^t with $R_2^0 = Q_2^0 = I_2$ degenerates to the pair of lines $I_1 I_2 \cup I_3 I_4$ (and $(\Gamma^0)^t$ has the point $I_1 I_2 \cap I_3 I_4$ as a component), and the conic touching $IR_2'^t$ with $R_2'^0 \neq Q_2^0$ has a smooth limit. This describes

$$X^0 = X_1^0 \cup X_2^0,$$
$$X_1^0 \text{ a conic},$$
$$X_2^0 = I_1 I_2 \cap I_3 I_4,$$

in Case (I.b). If we are in Situation (I.c), say

$$R_2^0 = Q_2^0 = I_2$$
$$R_2'^0 = Q_2'^0 = I_3$$

then both components of Γ' degenerate, and

$$X^0 = X_1^0 \cup X_2^0,$$
$$X_1^0 = I_1 I_2 \cap I_3 I_4.$$
$$X_2^0 = I_1 I_3 \cap I_2 I_4. \quad \text{Q.E.D.}$$

Remark 9.10. The situation can be clarified as follows. Consider

$$h F \to C \times C, \qquad h(F_i) = \Lambda_i$$

defined by

$$h(P, L_1, L_2) = (R_1, R_2)$$

where $L^i \cap C = \{P, R_i\}$. It can be seen that the twist-map

$$\imath : C \times C \to C \times C, \qquad \imath(R_1, R_2) = (R_2, R_1)$$

has the property

$$\imath \Lambda_i = \Lambda_i, \qquad i \in \{1, 2\}.$$

This gives a geometric explanation of the fact that each of the maps $f: F_i \to \Gamma_i$ has degree at least two: if (P, L_1, L_2) gives the chord $R_1 R_2 \in \Gamma_i$, then there exists a point $P' \in C$ and (P', L'_1, L'_2) giving the same chord $(R_2, R_1) \in L_i$ (see Figure 9.3). Moreover it can be proved that $F_i \to \Lambda_i$ has degree 2 if and only if Γ_i is a line. Starting with some point in I, and some conic in $\{D_1, D_2\}$ we produce chords through the points in I in 16 ways; it turns out that the number of different chords is either $8 = 4 + 4$ (in Case I.a) or $6 = 4 + 2$ (in case I.b) or $4 = 2 + 2$ (in Case I.c).

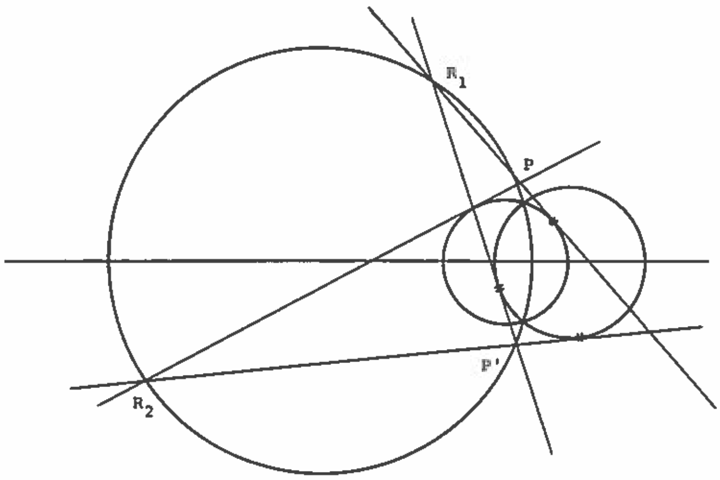


Figure 9.3

Remark 9.11. We describe the locus X in several degenerated cases (details are left to the reader). We still consider $I \subset \mathbb{P}^2$, $\# I = 4$, no three points on a line.

(9.11.1) The conics $C = D_1$ coincide, and $D_2 \neq C$ is a conic; in this case $X = D_2$ (counted twice).

(9.11.2) The conics $D_1 = D_2$ coincide and $D_1 \neq C$ is a conic; in this case $X = C \cup X_2$ (and $X_2 = D_1$ if the Poncelet construction closes for C, D_1 with $n = 3$).

(9.11.3) The conics $C \neq D_1$ are different, and D_2 is a pair of lines, say $D_2 = I_1 I_2 \cup I_3 I_4$ (and a line L is said to be tangent to D_2 if it contains the point $Q_2 = I_1 I_2 \cap I_3 I_4$). In this case $X_1 = X_2$, a conic in the same pencil. Let $I_1 R_2$ be constructed as before ($L_1 \cap D_1 = \{I_1\}$, $L_1 \cap C = \{P, I_1\}$, $L_2 = PQ_2$, $L_2 \cap C = \{P, R_2\}$), then X touches $I_1 R_2$ in I_1. It can happen that $X = D_1$ (iff $Q_2 P$ is tangent to C, in that case $R_2 = P$, etc.).

(9.11.4) We have a conic C, and two pairs of lines, say

$$D_1 = I_1 I_2 \cup I_3 I_4, \quad Q_1 = I_1 I_2 \cap I_3 I_4$$

$$D_2 = I_1 I_3 \cup I_2 I_4, \quad Q_2 = I_1 I_3 \cap I_2 I_4;$$

in this case X is the point

$$x := I_1 I_4 \cap I_2 I_3$$

(because the line $Q_1 Q_2$ is the polar line of x with respect to C, and this case is exactly the special case 8.5 i).

In all these cases it is obvious in which way a general situation degenerating to one of these cases gives a degeneration of the enveloping curve in question.

Remark 9.12. We have (in Situation I) that

$$F_1 \cap F_2 = \{I'_1, I'_2\ , I'_3, I'_4\},$$

and on the complement we have

$$f^0 : F - \{I'_1, I'_2, I'_3, I'_4\} \to \Gamma \subset (\mathbb{P}^2)^*;$$

extending to

$$f_1 : F_1 \to \Gamma \quad \text{and} \quad f_2 : F_2 \to \Gamma.$$

We show that in Case (I.a) the morphism f^0 cannot be extended on F, by showing that for $I'_i \in F_1 \cap F_2$ we have

$$f_1(I'_i) \neq f_2(I'_i), \qquad 1 \leq i \leq 4.$$

In fact $I'_i = (I_i, L_1, L_2)$ (cf. Figure 9.4) gives a chord S, this touches X_1 and X_2 in different points (because X_1 and X_2 have no points in common outside I, and

Figure 9.4

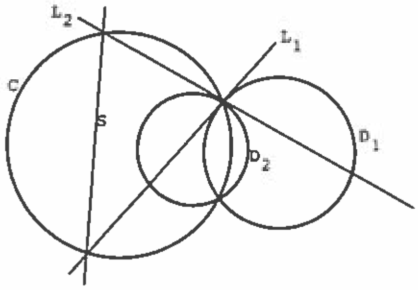
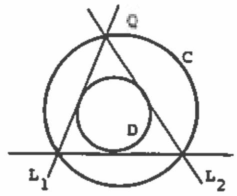


Figure 9.5

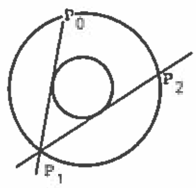


in Case (I.a) the chord S does not contain points in I), thus it corresponds to two different points of Γ. Note that X_1 and X_2 do not meet in the Cases (I.b) and (I.c) so in those cases g^0 does not extend.

Remark 9.13. The situation $D_1 = D_2 = D \neq C$ as we have seen in (9.11.2) can be considered in two ways. Let Y be the set of (unordered) pairs $\{L_1, L_2\}$ (cf. Figure 9.5) so that $Q = L_1 \cap L_2$, $Q \in C$, and these are the lines from Q tangent to D (and hence $L_1 = L_2$ iff $Q \in C \cap D$). The chords $R_1 R_2$ produced in this way envelop the curve X_2. We have

$$C \cong Y \cong X_2.$$

Figure 9.6



We can also start with $P_0 \in C$ and begin with the Poncelet traverse construction (cf. Figure 9.6): P_0, L_0, P_1, L_1, P_2. As there are two choices $(P_0, L_0) \in E = E(C, D)$ for a given $P_0 \in C_0$, $P_0 \notin D$, this can be done in two ways which however both give a point on X_2; in fact the map

$$E \to X_2, \quad (P_0, L_0) \mapsto P_0 P_0 P_2,$$

obtained in this way is 2:1. In fact, $P_0 P_2$ is the chord constructed from P_1 if we use the method described above. This second way of describing X_2 in case $D_1 = D_2$ comes nearest to the one we find in [Jacobi, 1828].

Remark 9.14. In Situation I, Case (I.a) the conics X_1 and X_2 described in (9.4.I) have the following properties. Choosing $P \in C \cap D_1 \cap D_2$ we obtain a chord S (see

Figure 9.7

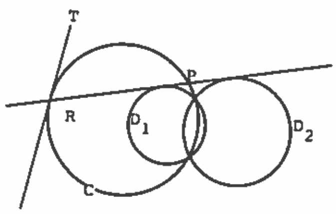


Figure 9.4); this chord touches X_1 and X_2 in two different points (and in this way we construct the 4 different lines in $X_1^* \cap X_2^*$). Let $L \in D_1^* \cap D_2^*$ and $P \in L \cap C$ (there are 8 such choices possible), let $L \cap C = \{P, R\}$, and let T be the tangent to C at R (see Figure 9.7); this line T touches one of the components X_1, X_2, and in this way we construct the 8 common tangents of C and X_1 and of C and X_2.

10. A curve enveloping a family of chords of a conic (Situation II)

In this section we study Situation II, i.e. C, D_1, $D_2 \subset \mathbb{P}^2$ are three different conics in general position, and we determine some properties of the curves $X \subset \mathbb{P}^2$ and $\Gamma \subset \mathbb{P}^2$ (defined in the construction in 8.3).

We choose the situation so that

$$C \cap D_1 \cap D_2 = \emptyset;$$

in this case $F = F(C, D_1, D_2)$ is a smooth irreducible curve of genus 5 (cf. Lemma 9.2.II); note that $C \cap D_1 \cap D_2 = \emptyset$ is an open condition, and we can choose C, D_1 and D_2 so that

$C \cap D_2 \cap D_2 = \emptyset$.

Proposition 10.1 *(in Situation II; cf. Lemma 9.6 and Lemma 9.8 for the case of Situation I). Let C, D_1, D_2 be in general position; then the morphisms $f: F \to \Gamma$ and $g: F \to X$ are birational and $\Gamma \subset (\mathbb{P}^2)^*$ is a curve of degree* 8.

Proof. We show first that for C, D_1, D_2 in general position, the degree $f: F \to \Gamma$ equals to one (and hence $\deg(g: F \to X) = 1$). This can be seen by a computation; we prefer a geometric argument.

We consider C, D_1 and $D_2^{(0)}$ in Situation I, Case (I.a). The morphism $F_1^{(0)} \to \Gamma_1^{(0)}$ has degree 2, we choose a general $R_1\ R_2 \in \Gamma_1^{(0)}$; in that case we have two different points on F mapping onto $R_1\ R_2$, say

$$(P, L_1, L_2) \in F_1, \quad L_i \cap C = \{P, R_i\}, \quad L_1 \cap D_1 = Q_1, \quad L_2 \cap D_2^{(0)} = Q_2$$

and

$$(S, M_1, M_2) \in F_1, \quad M_i \cap C = \{P, R_i\}, \quad M_1 \cap D_2^{(0)} = T_2, \quad M_2 \cap D_1 = T_1$$

with

$$\mathrm{P} \neq S, \quad R_2 \neq Q_2 \neq P.$$

We keep C, D_1, P, R_1, Q_1, Q_2 fixed, and we consider all conics $D_2^{(t)}$ touching L_2 in Q_2 (a family of dimension three). For all choices of t, we still have $(P,$

$L_1, L_2) \in F^{(t)}$, mapping onto $R_1\, R_2 \in \Gamma^{(t)}$. For general choice of $D_2^{(t)}$ (still touching L_2 in Q_2) we have $C \cap D_1 \cap D_2^{(t)} = \emptyset$ (hence $F^{(t)}$ is irreducible), and M_1 does not touch $D_2^{(t)}$. This shows that for general choice of $D_2^{(t)}$ there is exactly one point on $F^{(t)}$ mapping onto $R_1\, R_2 \in \Gamma^{(t)}$, and we conclude that $\deg(F^{(t)} \to \Gamma^{(t)}) = 1$ in such a case.

If C, D_1 and D_2 are in general position and $R_1 \in C$ is in general position, the 8 ways to produce $R_2 \in C$ with $R_1\, R_2 \in \Gamma$ give 8 different points of Γ (this follows from the argument above). Thus a general line R_1 touching C^* intersects Γ in a points. Because $C^* \not\subset \Gamma$ this shows that the degree of Γ equals 8. Q.E.D.

Remark 10.2. We define what we assume for $C, D_1, D_2 \subset \mathbb{P}^2$ to be the property of being "in general position", and we indicate some special positions for P, L_1 and L_2. We say C, D_1, D_2 are in general position if:

- $C \cap D_1 \cap D_2 = \emptyset$,
- the conclusion in 10.1 holds,

and if, moreover, the following conditions are satisfied (they will be referred to later by the codes indicated in the margin):

(a.1) If $L_1 \in C^* \cap D_1^*$, with $P \in L_1 \cap C$, and L_2 is a tangent through P to D_1 with $L_2 \cap D_2 = Q_2$, then

$$L_1 \cap D_1 =: Q_1 \neq P = R_1 \quad \text{and} \quad L_2 \cap C = \{P, R_2\} \quad \text{with } R_2 \neq Q_2 \neq P$$

(see Figure 10.1);

(a.2) The same condition as (a.1) with 1 and 2 interchanged.

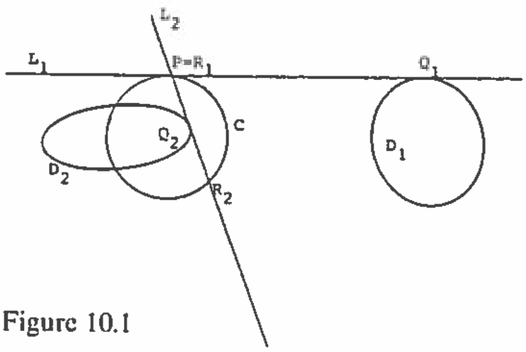


Figure 10.1

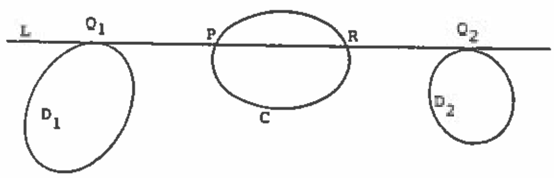


Figure 10.2

(b) If $L \in D_1^* \cap D_2^*$ (and the symbols as in Figure 10.2), then $P \neq R = R_1 = R_2$ (this is the same as $C^* \cap D_1^* \cap D_2^* = \emptyset$), and $P \neq Q_1 \neq R$ and $P \neq Q_2 \neq R$ and the pair (Q_1, Q_2) is not in harmonic position relative to (P, R); (there are 2 choices for P and R).

(c.1) If $P \in C \cap D_1$, and the other symbols as given in the Figure 10.3, then $P \neq R_1 \neq R_2 \neq P$, $R = Q_1$, and $P \neq Q_2 \neq R_2$.

(c.2) The same conditions as (c.1) with 1 and 2 interchanged.

Figure 10.3

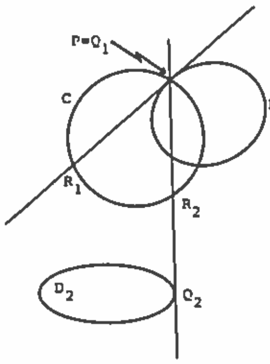
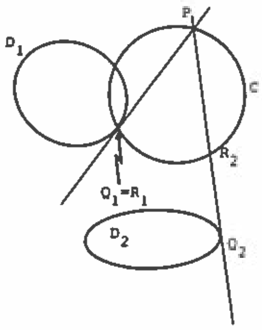


Figure 10.4



(d.1) If, with the symbols as in Figure 10.4, $P \neq R_1 \neq R_2 \neq P$, $Q_1 = R_1$, then $P \neq Q_2 \neq R_2$.

(d.2) The same condition as (d.1) with 1 and 2 interchanged.

Let $(P, L_1, L_2) \in F$ be a smooth point of F (in the case of general position all points of F are smooth by 9.2.II); then we define $y \in \mathbb{P}^2$ and $x \in X \subset \mathbb{P}^2$ in the following way; take the open set F^0 of points in F such that the points P, R_1, R_2, Q_1, Q_2 are mutually different, and construct for such positions the points y and x as indicated in the construction of the points G and H in 8.3, namely $Q_1 R_2 \cap Q_2 R_1 = y$, and $R_1 R_2 \cap P y = x$. The maps

$$G: F^0 \to \mathbb{P}^2 \quad \text{and} \quad H: F^0 \to \mathbb{P}^2$$

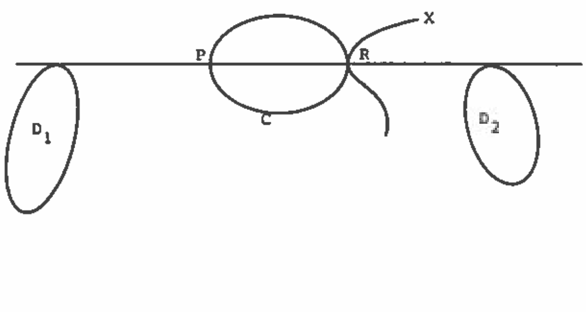
can be extended uniquely to the set of smooth points of F (e.g. in situation (b) above, $R_1 Q_2 = R_2 Q_1$, so the intersection of these two lines is not defined, but our construction still determines a unique y and x for this situation).

Lemma 10.3. *Assume C, D_1, D_2 in general position. If $(P, L_1, L_2) \in F$ then either Proposition 8.4 applies (and $x \notin C$), or we are in one of the cases (a.1), ..., (d.2). In these cases y and $x \in X$ are as follows:*

(a.1) $y = R_2$ and $x \notin C$;

(b) $x = R$, the curve X is smooth at x and it touches C at x (see Figure 10.5);

Figure 10.5



(c.1) $y = Q_2$ and $x = R_2$;

(d.1) $y = x = Q_1$.

Proof. It is easy to see that all cases described give all possibilities; we omit the proof.

Assume situation (a.1). After a suitable transformation of $\mathbb{P}^2$ we obtain the following situation (see Figure 10.6):

$$C: aX^2 + Y(Y-1) = 0, \qquad a \neq 0,$$

$$L_1 : Y = 0,$$

$$Q_2 = (0:1:c), \qquad c \neq 1.$$

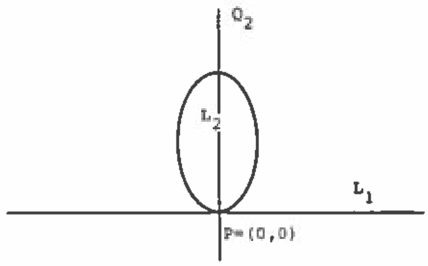


Figure 10.6

Because $P \neq Q_1$ and $P \neq Q_2$ we can apply Lemma 8.1 in both cases (to L_1 and L_2); for $R = k[\varepsilon]/(e^2)$ and

$$P_\varepsilon = (\varepsilon, 0).$$

we compute that $R_1 = (0, 0)$ deforms to

$$R_{1,\varepsilon} = (-\varepsilon, 0),$$

and that L_2 $X = 0$ and $R_2 = (0, 1)$ deform to

$$L_{2,\varepsilon}: X + \varepsilon c\, Y - \varepsilon Z = 0$$

$$R_{2,\varepsilon} = (\varepsilon(1-c), 1);$$

thus $\gamma = R_1 R_2$ deforms to

$$\gamma_\varepsilon : X + \varepsilon\, Y(c-2) + \varepsilon Z = 0.$$

In dual coordinates γ is the point $(1:0:0) \in (\mathbb{P}^2)^*$ and $\gamma_\varepsilon = (1:\varepsilon(c-2):\varepsilon)$, so this deformation is on the line $Y' + (2-c)Z' = 0$ (in dual coordinates) and we conclude

$$x = (0:1:2-c)$$

thus $x \notin C$.

For case (b) we dualize, and we choose coordinates such that:

p: $\quad Y, \quad l=(0,0)$

r: $\quad X$

C^*: $\quad X^2+2X+1+2Y+Y^2$

D_i^*: $\quad X-\gamma_i Y+\ldots$, parametrization $(\gamma_i t+\delta_i t^2+\ldots, t)$;

let $a^3=0$, consider $k[a]/(a^3)$, and let

$$u^{(a)}=(-1+a, -a^2/2),$$

$$p^{(a)}: 2aX+(2-a^2)Y+2a-a^2$$

we solve:

$$t_i=-a+a^2(1/2+\gamma_i)$$

and we compute

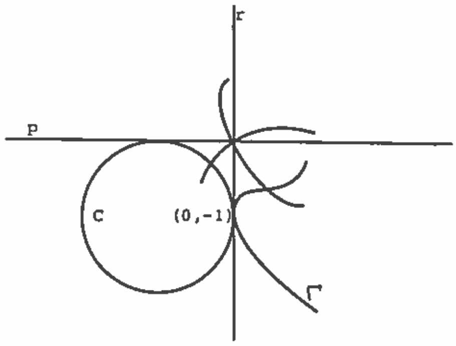
$$r_i^{(a)}: X+[\gamma_i a+a^2(-\gamma_i-\gamma_i^2-\delta_i)]\,Y+\gamma_i a+a^2(-\gamma_i-2\gamma_i^2-\delta_i);$$

these two lines intersect at

$$(a^2\gamma_1\gamma_2, -1+a(\gamma_1+\gamma_2))$$

(we use $\gamma_1\neq\gamma_2$). If P, R, Q_1, Q_2 are not harmonic on L, i.e. $\gamma_1+\gamma_2\neq 0$, the curve $\Gamma\subset(\mathbb{P}^2)^*$ is smooth at $(0,-1)$ with the property that the line $X=0$ intersects with multiplicity 2, and all other lines through $(0,-1)$ intersect Γ with multiplicity 1 (cf. Figure 10.7). Thus $\Gamma^*=X$ passes through R with the same tangent direction as C in that point.

Figure 10.7



Cases (c.1) and (d.1) are proved by specialization as in the proof of Lemma 9.7. Q.E.D.

Theorem 10.4.II. *Suppose C, D_1, $D_2\subset\mathbb{P}^2$ in general position (i.e. the conditions in 10.2 are fulfilled). Then:*

a) $\Gamma\subset(\mathbb{P}^2)^*$, and Γ has no cusps,

b) $X\subset\mathbb{P}^2$ has degree 24; in case (b) the intersection multiplicity of X and C at

R equals 2 (see Figure 10.5), in the cases (c.1), (c.2), (d.1) and (d.2) the curve X is smooth at x, and X intersects C transversally at that point (see Figures 10.8 and 10.9).

Figure 10.8

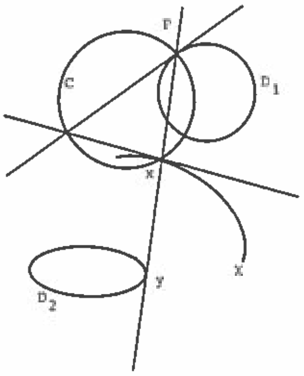
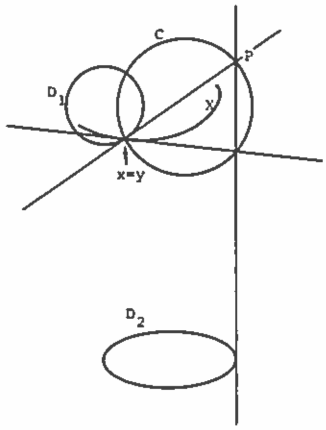


Figure 10.9



Proof: In situation (b) we have 8 choices for P, in situations (c.1) and (c.2) there are 16 possibilities, and for (d.1) and (d.2) there are 16 possibilities. Thus the total number of intersections (counted with multiplicities) of X and C is at *least*

$$8 \cdot 2 + 16 + 16 = 48;$$

however, the genus of F equals 5, thus the sum of the multiplicities r_i of the singularities of Γ is given by

$$(8-1)(8-2)/2 - 5 = 16 = [\sum r_i(r_i - 1)]/2$$

(cf. [Walkers 1950], VI 5.2, 5.5). Thus the degree of the dual of Γ,

$$\deg(\Gamma^*) \leq 8 \cdot 7 - 2.16 = 24,$$

and the equality holds iff Γ has no cusps (by the Plücker formulas, cf. [Walker 1950], IV.6.3); from this we conclude that the total number of intersections of $X = \Gamma^*$ and C is at *most* $24 \cdot 2 = 48$ (by the theorem of Bezout, cf. [Walker 1950], IV.5.2, Theorem 5.4). From these two inequalities the statements in the theorem follow. Q.E.D.

Remark 10.5. By local computations we could have proved that the curve Γ has no cusps in Situation II. However, by such computations one cannot find singularities like nodes and higher singularities.

Remark 10.6. Suppose C^t, D_1^t, $D_2^t \subset \mathbb{P}^2$ are conics depending on a parameter, such that for $t \neq 0$ we have Situation II and for $t = 0$ we have Situation I. We have seen that $\Gamma^t \subset (\mathbb{P}^2)^*$ has degree 8 for general t, and it specializes to two double conics

$$(\Gamma^0)_{\text{red}} = \Gamma_1 \cup \Gamma_2$$

(as a scheme Γ^0 still has degree 8, and we can take the underlying reduced scheme $(\Gamma^0)_{red}$). Now

$$(\lim_{t\to 0} \Gamma^t)^* \neq \lim_{t\to 0}(\Gamma^t)^*)!!$$

There should exist a "good theory" for dualizing plane curve (with singularities, with multiple components) such that (with that new definition of a dual curve) specialization and dualizing commute. Note that for t general, $X^t=(\Gamma^t)^*$ has degree 24; this curve specializes to a curve X^0 such that

$$X^0=2X_0\cup 2X_1\cup\left(\bigcup_{j=1}^{8} T_j\right)\cup\left(\bigcup_{k=1}^{4} 2S_k\right),$$

where the lines T_j and S_k are as constructed in Remark 9.14. We hope that the situation just explained may stimulate a search for a "good theory" for dualizing arbitrary plane curves.

Remark 10.7. We have used the word "component" of a curve; this may lead to confusion. Suppose given an algebraic curve, say $Y\subset\mathbb{P}^2$, defined by an equation

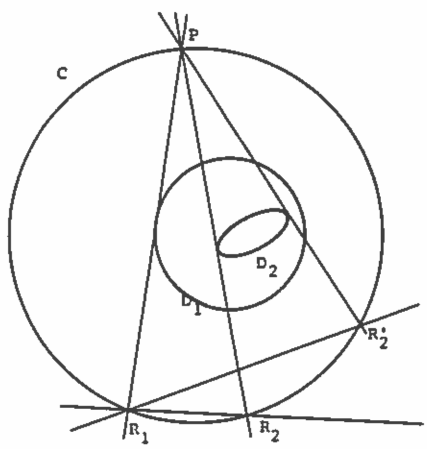
$$F=0,$$

where F is a polynomial with coefficients in $\mathbb{R}$. Then that algebraic curve may be irreducible, in the sense of algebraic geometry, and that is equivalent to saying that F is an irreducible polynomial, if considered over the complex numbers (sometimes we say that Y is absolutely irreducible). The components of a curve Y (over $\mathbb{C}$) are the irreducible curves contained in Y; they correspond to the irreducible factors (over $\mathbb{C}$) of F. However, if we draw pictures (over $\mathbb{R}$) of algebraic curves it may happen that the real points of the curve form disconnected parts. For example it is easy to give C, D_1, $D_2\subset\mathbb{P}^2$ defined over $\mathbb{R}$ in Situation II such that X has several topological components of $X(\mathbb{R})$; we have then:

- X has one component (is irreducible as an algebraic curve)
- $X(\mathbb{R})$ has several topological components.

Consequently it is dangerous to decide (ir)reducibility on the grounds of real-topological pictures. Example (see Figure 10.10);

Figure 10.10



$x \in X(\mathbb{R}) \cap R_1 R_2$ and $x' \in X(\mathbb{R}) \cap R_1 R_2'$ are on different topological components of $X(\mathbb{R})$, we see $X(\mathbb{R})$ has more than one (topological) component, and being in Situation II, the algebraic curve X is irreducible.

11. A comparison of the theorems and the proofs

11.1 In connection with Griffiths' proof of the closure theorem it has been suggested that the ideas of the modern proof must have been present in the older studies of Poncelet, Jacobi and even Fuss. Griffiths wrote:

> "It is interesting to note that the result we shall prove [the closure theorem] will be *equivalent* to the addition law for an elliptic integral, so that the early somewhat complicated proofs of the Poncelet theorem must have amounted to synthetic derivations of this addition formula, presumably in the same way in which the addition formula for the sine function may be derived by drawing pictures." [Griffiths, 1976] p. 345.

And Griffiths and Harris:

> "... the Poncelet theorem and the addition theorem are essentially equivalent, so that at least in principle Poncelet gave a synthetic derivation of the group law on an elliptic curve." [Griffiths, Harris, 1977] p. 145.

And Mazur:

> "Griffiths pointed out to me that the data of the classical Poncelet theorem (–) provides one with an elliptic curve and a point of order n on that elliptic curve. (As was known, in effect, to Jacobi, see [ref. to [Griffiths, 1976] par. 1d].) But judging from hints given in [ref. to [Dörrie, 1965]], the mathematician Nicolaus Fuss (–) may have found rational parametrizations of *Poncelet* quadrilaterals, pentagons, hexagons, heptagons and octagons (–)." [Mazur 1977] p. 108.

The closure property and the occurrence of elliptic functions, elliptic curves and addition laws in the proofs indeed suggest a strong affinity between these results. On the other hand, by means of expressions like "must have amounted", "in principle", "in effect", the quoted authors allow for (further unspecified) differences.

The question in how far later ideas can be legitimately recognized in earlier mathematical work is indeed a very difficult one. It is also a question with which both the historian of mathematics and the mathematician are often confronted. The historian will perhaps tend to stress the differences between the historical and the modern versions of mathematical theories, while the mathematician may want to bring out the similarities; and the final conclusion on such questions will always leave room for personal judgement. However, we feel that it is not entirely a question of taste, and that it is useful in these cases to specify and clarify both the differences and the similarities before coming to conclusions. In the present section we will do that for the closure theorem, the "main theorem" and the "general theorem", comparing the studies of Poncelet, Jacobi and Griffiths, and concentrating on the theorems themselves, the extent to their validity, the objects introduced in the proofs, and the styles.

11.2 We first recall the relation between the two theorems. The main theorem is the special case of the general theorem in which the number of vertices of the traverse is equal to three. Poncelet could not prove the closure theorem directly; he needed the general theorem. And in his proof by induction of that theorem he needed the main theorem. Jacobi did not use the general theorem to prove the closure theorem; he derived both by applying the theory of elliptic functions. His proof of the general theorem is not by induction but direct; he had no need to prove the main theorem first and he did not deal with the case $n=3$ separately. It seems that he discussed the general theorem primarily because Poncelet had stated it and because he could prove it easily. As the title of his article indicates, Jacobi was first of all interested in the closure theorem. Griffiths discussed neither the general nor the main theorem.

11.3 Poncelet proved his theorems for circles and used projection and the principle of continuity to pronounce them valid for all conics, by which, as we have explained in Section 3.2, he meant conics in the real plane. Jacobi proved the closure theorem for circles without real intersections, the one lying inside the other. He mentioned the projection theorem as a means of generalising the theorems, but he added the condition that the circles should have not more than two real intersections. That is, he did not rely on Poncelet's principle of continuity and he did not consider projections with imaginary centres. He must therefore have considered his theorems as proved for real conics without real intersections. Griffiths proved the closure theorem for pairs of smooth conics in $\mathbb{P}^2(\mathbb{C})$ with four distinct intersections. Poncelet studied the case that the two conics are concentric circles, that is, that they are tangent to each other in the circular points. He did not in general study cases of tangency; neither did Jacobi or Griffiths. (A study of the cases of tangency is supplied in the present article, Section 7.14.)

11.4 The central object in the "main theorem" is the envelope X of the family of chords arising in the manner discussed in Sections 4.3 and 8.3. Poncelet proved the closure theorem by showing that, in the case of closure of one interscribed traverse between C and D, the envelope X coincides with D. Thus it plays an essential role in Poncelet's proof. The envelope occurs in Jacobi's proof of the general theorem, but not in his proof of the closure theorem. It does not occur in Griffiths' proof. Neither Poncelet nor Jacobi discussed the behaviour of this envelope in the case that the conics do not belong to the same pencil. (This behaviour, which is interesting because it supplies a useful example of non commutativity of limits and dualizing, is discussed in Section 10 of the present article. In fact the complicated nature of that envelope in the general case made us for a long time suspicious of Poncelet's proof of the main theorem with its ad hoc use of infinitesimal motions; ultimately Jacobi's proof convinced us – we should have looked at his article earlier.)

11.5 Both Jacobi's and Griffiths' proofs of the closure theorem involve an addition law; In Jacobi's case the addition law of elliptic functions, in Griffiths' case the additive structure of an elliptic curve. Griffiths considers the object

$$E=E(C,D)=\{(P,L)\,|\,P\in C, L\in D^*, P\in L\}$$

and proves that it has the structure of a smooth elliptic curve (Lemma 7.1). Constructing a Poncelet traverse corresponds to repeated application of a mapping $\alpha: E\rightarrow E$, and it is shown that this mapping corresponds to a translation by an element t (determined by C and D in E^2) in the group structure on E,

$$\alpha(x)=x+t. \tag{11.1}$$

Jacobi studied the function $W:\mathbb{R}\rightarrow\mathbb{R}$ which describes the transition, on the outer circle C, from one vertex of a Poncelet traverse to the next, where C is parametrized by the central angle. He gave that function explicitly, showing that

$$W(\mathrm{am}\,u)=\mathrm{am}(u+c) \tag{11.2}$$

for some constant c; he then used the theory of elliptic functions to complete the proof.

Both proofs relate the closure property to the operation of adding a constant. Indeed it is easy to see that, if we introduce the projection π: $E\rightarrow C$, $\pi(P, L)=P$, Jacobi's W is the projection of α on C. This relation between the proofs strongly suggests a similarity. Most mathematicians, after having read Griffiths' proof, will experience a sense of recognition when studying Jacobi's proof. Or they will, when reading the latter proof unprepared, start searching for the elliptic curve which, to them, is the obvious structure suggested by Jacobi's analytic formulas.

However, the question remains whether, because of these similarities, one can say that the elliptic curve E is present in Jacobi's proof. Jacobi himself did not explicitly introduce the structure of an elliptic curve. He proved Formula 11.2 constructively by direct computation. Griffiths' proof, on the contrary, is not constructive; it first recognizes E as an elliptic curve and then uses existence theorems to conclude that there is an additive structure. The crucial argument in recognising E as an elliptic curve lies in its behaviour over the branching points $C\cap D$. Jacobi, working with non-intersecting circles in the real plane, did not discuss this branching process in any way; in particular he did not discuss an analogon of the fact that E is a double covering of C.

Once more, consider the set E introduced in the modern proof. It is a geometrical object, consisting of pairs of points and lines, and it serves as a model of an elliptic curve. But it does not offer a natural interpretation of addition within the elliptic structure; it does not suggest a construction for finding the sum of two elements (P, L) and (Q, M) of E. The only addition that can be geometrically interpreted in this model is the translation $x \to x+t$, performed by following the Poncelet traverse. Jacobi, on the other hand, did offer an explicit geometrical con- stuction of the addition

$$(\mathrm{am}\,u, \mathrm{am}\,v) \to \mathrm{am}(u+v)$$

(cf. Section 5.4). This suggests, curiously enough, that *if* there would be an elliptic curve structure implicit in Jacobi's argument, it would have a more versatile model than the E of the modern proof. Let us recall Jacobi's construction. He fixed a modulus k for the elliptic function am. He then considered (cf. Figure 5.3) a circle C with radius R and centre M on the axis MO. He introduced the pencil of circles D with radii r and centres m on MO, with $mM = a$, where a and r are linked by

$$k^2 = 4aR/[(R+a)^2 - r^2].$$

He then adjusted to every point P on C (with $\angle OMP = 2\phi$) a circle D_P from the pencil, determined by

$$\cos\phi = r/(R+a), \qquad 1 - k^2\sin^2\phi = (R-a)^2/(R+a)^2.$$

His construction may now be interpreted as follows: Given points P and Q on C (with $\angle OMP = 2\phi = 2\mathrm{am}\,u$ and $\angle OMQ = 2\psi = 2\mathrm{am}\,v$), draw the tangent from P counterclockwise to D_Q. Let the second intersection of that tangent with C be R, with $\angle OMR = 2\chi$, then

$$\chi = \mathrm{am}(u+v).$$

In other words, Jacobi geometrically defined an additive structure on C, which indeed coincides with the additive structure on one (real) branch of E lying over C. If we introduce the possibility of drawing tangents clockwise as well, we may also incorporate the other branch as follows. Take

$$M = \{(P, \delta_P) \mid P \in C, \delta_P \in (-1, 1)\}$$

and define the addition

$$(P, \delta_P) + (Q, \delta_Q) = (R, \delta_R)$$

by:

- $R \in C$,
- PR tangent to D_Q, taken counterclockwise if δ_P is 1, clockwise if $\delta_P = -1$,
- $\delta_R = \delta_P \cdot \delta_Q$.

M is then the model of the (real) elliptic curve.

This exercise brings us rather far from what Jacobi did; it is meant to show that if there were an elliptic curve structure underlying Jacobi's arguments, its model would be quite different from, and indeed more versatile than E.

However, Jacobi devoted no attention at all to the specifically structural aspects such as the branching points and the double covering. In the modern approach attention to these features is essential because from them we conclude that E has the structure of an elliptic curve. For Jacobi this is not necessary; from the outset he had all the information about the structure because he started from the elliptic function am.

Surveying the aspects of similarity and difference between the two proofs as discussed above, we feel that there is no evidence that Jacobi consciously recognized the structure of an elliptic curve in the analytical and geometrical objects he was studying. We also feel that the similarities are not strong enough to justify a statement that "implicitly" or "essentially" Jacobi was studying such a structure.

11.6 There are marked differences in style between the three proofs. Griffiths' proof proceeds in the way familiar to the present-day mathematician: a special mathematical structure is recognized in a problem situation; the general theory about that structure is then fully exploited to settle the problem. Poncelet dealt with real figures and their motions, not with abstract structures. He avoided analytical arguments, he relied on an intuitive understanding of motion and in particular of infinitesimal motion, and he consciously stretched the mathematical rules of inference by his principle of continuity. Jacobi also dealt with real figures; he derived explicit analytical relations embodied in the figure and then applied the analytical theories and techniques to settle the question.

These differences of style should not be underestimated. Indeed the conception, both of the mathematical objects and of the nature of mathematical reasoning, in the studies of Poncelet and Jacobi, contrasts so strongly with the modern conception that, in preparing this article, we have occasionally felt the confrontation with early nineteenth-century mathematical style as a real culture shock.

11.7 Contradictory experiences, then, are involved in comparing the theorems and the proofs. There are marked differences of style content; nevertheless there is an essential similarity.

It seems to us that that contradiction should be recognized; it should not be blurred in cheap synthesis or compromise. The recognition of similarities between mathematical studies that are so different as the three we are discussing, relates to a willingness to consider mathematicians of the past essentially as colleagues engaged in the same enterprise of mathematical research. This willingness among mathematicians is, we feel, most important. It provides an active audience for historical studies, and it ensures a link with the past (which may at some occasions even lead to new ideas and questions in modern research). For mathematics, as for individuals and nations, it is dangerous to lose or distort the memory of the past. The recognition of earlier mathematics as mathematics, despite great differences in style and content, and the willingness to consider past mathematicians as colleagues, keep the interest in the past awake. Historical study, stressing and articulating the differences, guards against the dangers of distorting the past.

Note

The present study originates from research undertaken by the second and the fourth author as final work for their mathematics degree at the Utrecht Mathematical Institute. That work was supervised by the other two authors, it was inspired by the remarks of Griffiths, Harris and Mazur quoted in Section 11.1 and it was completed in 1981. At that time already we decided that some of the findings of that study deserved to be made public, but that such a publication required considerable further work both in understanding the old proofs and in extending the modern results. That study was undertaken by the first and third authors and the combined results are presented here. A preliminary version of the present text was circulated as preprint of the Utrecht Mathematical Institute in November 1984.

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