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Modular curves and Poncelet polygons*

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0 Introduction

A *Poncelet-polygon* is a polygon in the projective plane $\mathbb{P}_2 = \mathbb{P}_2(\mathbb{C})$ (the base field always is $\mathbb{C}$) with its vertices on one smooth conic $D \subset \mathbb{P}_2$ while its sides touch another smooth conic C. If the polygon happens to be a n-gon, we call the conic C *n-inscribed into D*, and D *n-circumscribed about C*. If m divides n, we do *not* consider a m-gon a special kind of n-gon. The aim of this note is to compute the following numbers:

- The number of conics D in a general pencil $C_{(\lambda:\mu)}$, which are n-inscribed into a fixed conic C of this pencil,
- the number of conics D in a general pencil, which are n-circumscribed about a fixed conic C of this pencil,
- the number of projective equivalence classes of pairs C, D of conics (in general position, i.e. meeting in four distinct points) such that C is n-inscribed into D and D is m-circumscribed about C.

To formulate our results, we need the number

$$t(n) := \text{the number of primitive } n\text{-torsion points on an elliptic curve}.$$

Here we mean by a "primitive" n-torsion point some point, which is not torsion of any order smaller than n. Clearly the function $t(n)$ is multiplicative in the sense of

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number theory. So if $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$ is the prime factor decomposition of n then

$$t(n) = (p_1^2 - 1)p_1^{2(k_1-1)} \cdot \ldots \cdot (p_r^2 - 1)\, p_r^{2(k_r-1)} .$$

For example

$$t(n) := \begin{cases} n^2 - 1 & \text{if } n = p \text{ is an odd prime} \\ n^2 - 4 & \text{if } n = 2p \text{ with } p \text{ an odd prime} . \end{cases}$$

Unless $n = 2$, the number $t(n)$ is divisible by four. We abbreviate

$$c(n) = \tfrac{1}{4} \cdot t(n) .$$

Our results are

Theorems 1 and 2. *Each conic $C_{(\lambda:\mu)} = \lambda C + \mu D$ in a generic pencil (this means C and D intersect in four distinct points) is n-inscribed into $c(n)$ conics in this pencil, and n-circumscribed about twice that number of conics in the pencil.*

Of course, this number includes certain multiplicities, but for a general conic C in the pencil the multiplicities are one.

Theorem 3. *Each smooth conic C in the plane $\mathbb{P}_2$ is (up to projective equivalence) simultaneously n-inscribed and m-circumscribed about*

$$\begin{cases} 2 & \textit{if } m = n = 3 \\ \leq \tfrac{1}{3} \cdot c(m)c(n) & \textit{if } m \text{ or } n > 3 \end{cases}$$

conics D (meeting C in four distinct points), counted up to projective equivalence.

This number again contains multiplicities, but unfortunately here we cannot control them.

The proof of these facts consists of relating them to plane projective models of certain modular curves: It is well-known that the Poncelet-property depends on a torsion element in the elliptic curve, which is a double cover of C, branched over the four points of intersection of C with D, cf. [GH]. We only put one parameter into this situation and study torsion sections on rational elliptic surfaces, which are double covers of the plane. The image of the n-torsion curve on this surface is a curve $\Pi_n \in \mathbb{P}_2$, which is the birational image of the modular curve $X_{0,0}(n, 2)$ parametrizing isomorphism classes of

- elliptic curves with a level-2 structure
- and a primitive torsion-point of order n on this curve.

It is easy to see that

$$\text{degree}(\Pi_n) = c(n) .$$

The numbers mentioned are

- the intersection number of Π_n with a line,
- the intersection number of Π_n with a conic,
- the intersection number of Π_m with Π_n^*, the image of Π_n under a certain Cremona transform, which therefore is a curve of degree $2 \cdot c(n)$. This intersection number is divided by six, because of projectively equivalent situations.

1 The elliptic surface

We fix a general pencil $C_{(\lambda:\mu)}$ of conics in the plane $\mathbb{P}_2$. Here "general" means that the pencil has four base points. We call these base points P_0, P_1, P_2, P_3. They are in general position, so we can choose homogeneous coordinates (x_0, x_1, x_2) on $\mathbb{P}_2$ such that

$$P_0 = (1:1:1)\,, \quad P_1 = (-1:1:1)\,, \quad P_2 = (1:-1:1)\,, \quad P_3 = (1:1:-1)\,.$$

The pencil then consists of all conics

$$\lambda x_0^2 + \mu x_1^2 - (\lambda + \mu)x_2^2, \quad \lambda:\mu \in \mathbb{P}_1\,.$$

All these conics are smooth but for the three values

$$(\lambda:\mu) = (1:0), (0:1), (1:-1)\,,$$

for which the conic $C_{(\lambda:\mu)}$ splits into a pair $L_k, L_k', k = 1, 2, 3$, of lines. We denote by L_k the line in this pair containing P_0.

Each conic in the pencil is invariant under the group

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \begin{cases} \mathbb{P}_2 \to \mathbb{P}_2 \\ (x_0{:}x_1{:}x_2) \mapsto (\pm x_0 : \pm x_1 : \pm x_2)\,. \end{cases}$$

We denote by $E_{(\lambda:\mu)}$ the double cover of the conic $C_{(\lambda:\mu)}$ branched over the four points $P_0, P_1, P_2, P_3 \in C_{(\lambda:\mu)}$. If $C_{(\lambda:\mu)}$ is nondegenerate, then $E_{(\lambda:\mu)}$ is a smooth elliptic curve. We distinguish the point over P_0 as origin on $E_{(\lambda:\mu)}$. Then the three points over P_1, P_2, P_3 are the three non-trivial half-periods on $E_{(\lambda:\mu)}$. The chosen ordering P_0, P_1, P_2 of these three points defines an ordering of the three non-trivial half-periods on $E_{(\lambda:\mu)}$, i.e. a *level-2 structure on the elliptic curve* $E_{(\lambda:\mu)}$.

(1.1) *Given an elliptic curve E with a level-2 structure there is a unique curve $E_{(\lambda:\mu)}$ and an isomorphism $E \to E_{(\lambda:\mu)}$ compatible with level-2 structures. In this way we get an identification of the parameter curve $\mathbb{P}_1(\lambda:\mu)$ of our pencil with the modular curve X_2, which parametrizes elliptic curves with level-2 structure.*

Proof. We represent E as a double cover of some $\mathbb{P}_1$ with four branch points $e_0, e_1, e_2, e_3 \in \mathbb{P}_1$, the point over the branch point e_0 being the origin on E. We assume that the three other branch points are ordered in the way of the level-2 structure and denote the cross-ratio $[e_0, e_1, e_2, e_3]$ by α.

The cross-ratio $[P_0, P_1, P_2, P_3]_{C_{(\lambda:\mu)}}$ of the four points $P_0, \ldots, P_3$ on a conic $C_{(\lambda:\mu)}$ is computed as follows: Choose an auxiliary line $L \subset \mathbb{P}_2$, e.g. the line $x_0 = 0$. Project the three points P_1, P_2, P_3 from P_0 into this line to obtain the points $(0:1:1)$, $(0:1:0)$, and $(0:0:1)$. The tangent $T_{P_0}(C_{(\lambda:\mu)})$ of the conic $C_{(\lambda:\mu)}$ at the point P_0 is the line $\lambda x_0 + \mu x_1 - (\lambda + \mu)x_2 = 0$. It meets the line L in the point $(0:\lambda + \mu:\mu)$. So we find

$$\begin{aligned}
[P_0, P_1, P_2, P_3]_{C_{(\lambda:\mu)}} &= [(1:1), (1:0), (0:1), (\mu:-\lambda)]_{\mathbb{P}_1} \\
&= \left[1, 0, \infty, \frac{\mu}{\lambda + \mu}\right] \\
&= \frac{\lambda}{\mu}\,.
\end{aligned}$$

For $\frac{\lambda}{\mu} = \alpha$, and only for this value of $(\lambda : \mu)$, there is an isomorphism $\mathbb{P}_1 \to C_{(\lambda:\mu)}$ sending e_k to P_k, $k = 0, \ldots, 3$. It induces an isomorphism $E \to E_{(\lambda:\mu)}$, uniquely up to the covering involution. □

Our next aim is to combine all the elliptic curves $E_{(\lambda:\mu)}$ into one elliptic surface. As there is no universal elliptic curve with level-2 structure, this is possible only with a slight modification:

We fix one smooth conic C in our pencil $C_{(\lambda:\mu)}$. We denote by $\gamma: \mathbb{P}_1 \times \mathbb{P}_1 \to \mathbb{P}_2$ the double cover with branch locus C. Then $\gamma^{-1}(C_{(\lambda:\mu)})$ is isomorphic with the elliptic curve $E_{(\lambda:\mu)}$ for all conics $C_{(\lambda:\mu)} \neq C$.

We denote by $\sigma: \tilde{\mathbb{P}}_2 \to \mathbb{P}_2$ the blowing-up of the four base points $P_0, \ldots, P_3$ with $E_k = \sigma^{-1} P_k \subset \tilde{\mathbb{P}}_2$ the exceptional curve over $P_k, k = 0, \ldots, 3$.

The pull-back $\tilde{\gamma}: Y \to \tilde{\mathbb{P}}_2$ of γ to $\tilde{\mathbb{P}}_2$ is a double cover branched over $\tilde{C} + E_0 + \ldots + E_3$, the total transform of C under σ. The surface Y has four ordinary double points over the four intersections $\tilde{C} \cap E_k$, $k = 0, \ldots, 3$. We denote by $\tau_1: X \to Y$ the minimal desingularization of this surface. So we have the following diagram of maps:

$$\begin{array}{ccccc} X & \xrightarrow{\tau_1} & Y & \xrightarrow{\tau_2} & \mathbb{P}_1 \times \mathbb{P}_1 \\ & & \downarrow{\scriptstyle \tilde{\gamma}} & & \downarrow{\scriptstyle \gamma} \\ & & \tilde{\mathbb{P}}_2 & \xrightarrow{\sigma} & \mathbb{P}_2 \,. \end{array}$$

The pencil $C_{(\lambda:\mu)}$ lifts to $\tilde{\mathbb{P}}_2$ as a basepoint-free pencil of conics and to the rational surface X as an elliptic fibration. We denote by $F_{(\lambda:\mu)} \subset X$ the fibre in this fibration over the curve $C_{(\lambda:\mu)}$. These fibres are

- smooth elliptic isomorphic with $E_{(\lambda:\mu)}$, if $C_{(\lambda:\mu)}$ is smooth and different from the fixed conic C,
- of Kodaira-type $I_0^* = \tilde{D}_4$ if $C_{(\lambda:\mu)} = C$ (the central component in this fibre lies over C while the four other components $C_0, \ldots, C_3$ are the (-2)-curves resolving the double points of Y),
- of Kodaira-type I_2 if $C_{(\lambda:\mu)}$ is one of the three degenerate conics $L_k + L_k'$ in the pencil. (We denote these fibres by $D_k + D_k'$ with D_k the curve over L_k.)

The elliptic fibration $F_{(\lambda:\mu)}$ of X admits four sections $S_0, \ldots, S_3$ lying over the four exceptional curves $E_0, \ldots, E_3 \subset \tilde{\mathbb{P}}_2$. The images in $\mathbb{P}_2$ of these sections are the four base points $P_0, \ldots, P_3$. The section S_0 cuts out on each smooth fibre $F_{(\lambda:\mu)}$ the point over P_0, which we view as origin on the elliptic curve $F_{(\lambda:\mu)}$. So S_0 is the *zero-section*. The other three sections S_1, S_2, S_3 meet the smooth fibres $F_{(\lambda:\mu)}$ in the other three branch points of the covering $F_{(\lambda:\mu)} \to C_{(\lambda:\mu)}$. So these are *two-torsion sections*.

Translation by two-torsion sections defines a group action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the surface X. This covers the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on $\mathbb{P}_2$ described above, which leaves each conic in the pencil invariant.

A final remark: The surface X and its elliptic fibration depend on the chosen conic C. This dependance is unwanted, but inevitable. Fortunately everything interesting for us in the sequel will be independent of this choice.

2 The n-torsion curve Δ_n

Having specified a zero-section S_0 for the elliptic fibration on X we can talk about the points of order n on each smooth fibre $F_{(\lambda:\mu)}$. On X there is a closed algebraic curve T_n for each $n \in \mathbb{N}$, defined by two properties:

- on each smooth fibre T_n cuts out the n^2 points of order n,
- T_n does not contain any fibre components.

The curve T_n contains the zero-section S_0, and for n even it contains S_1, S_2, S_3 too. These two-torsion sections S_i do not meet other components of T_n on smooth fibres $F_{(\lambda:\mu)}$, nor in smooth points of singular fibres (since these smooth points form a one-dimensional Lie group [Ko, Theorem 9.1]). As the sections S_i do not pass through singularities of fibres, they form connected components of the curve T_n. We define

$$\Delta_n := \begin{cases} T_n \backslash S_0 \\ T_n \backslash (S_0 \cup \ldots \cup S_3) \end{cases} \text{for} \begin{cases} n \text{ odd} \\ n \text{ even} . \end{cases}$$

The aim of this section is to describe the curve Δ_n.

(2.1) *The curve Δ_n meets the I_0^*-fibre over C only on its (nonreduced) central curve, and not on the curves C_i, $i = 0, \ldots, 3$.*

Proof. We use stable reduction [BPV, III.10]. A neighbourhood of the I_0^*-fibre can be represented as the quotient of a smooth fibration by an involution, with $C_0, \ldots, C_3$ the images of the four fixed points of this involution blown up. These fixed points are just the two-torsion points on the central fibre of the stable reduction. Near the I_0^*-fibre, the curve Δ_n is the image of the curve of n-torsion points Δ_n' on the stable reduction, the two-torsion sections being removed. Now Δ_n' does not meet the central fibre of the stable reduction in points of order two, and its image $\Delta_n \subset Y$ does not meet the blow ups $C_0, \ldots, C_3$. □

It remains to describe Δ_n near the three singular fibres $F_k = D_k + D_k'$ of type I_2. Recall that $D_k \subset F_k$ is the component met by the zero-section S_0.

(2.2) *Near each of the two double points of F_k the n-torsion curve Δ_n decomposes locally into $\left[\frac{n-1}{2}\right]$ components $\Delta_n^{(j)}$, $j = 1, \ldots, \left[\frac{n-1}{2}\right]$, (not necessarily irreducible ones) of multiplicity* $\min\{2j, n-2j\}$. *The intersection numbers of $\Delta_n^{(j)}$ with D_k and D_k' are*

$$(\Delta_n^{(j)} \cdot D_k) = n - 2j, \qquad (\Delta_n^{(j)} \cdot D_k') = 2j .$$

Proof. By [Ko, Theorem 9.1], the smooth points on F_k form a group $F_k^\# \simeq \mathbb{C}^* \times \mathbb{Z}_2$. Its connected component of the origin belongs to D_k and contains a cyclic subgroup $\mathbb{Z}_n$. This group $\mathbb{Z}_n$ extends to a group of sections in a neighborhood U of F_k, cf. [BPV, V.9]. We form the quotient

$$q: U \to Q$$

by the action of this group. The two singularities of F_k go to two quotient singularities of type A_{n-1}, D_k goes n-to-one onto a rational curve E and D_k' goes n-to-one onto another rational curve E' meeting E in the two singularities.

Next we resolve the two A_{n-1}-singularities of Q by inserting a string of $n-1$ rational curves over each of them:

$$r: R \to Q .$$

In R we have a cycle of $2n$ rational curves. We call them $E_0, \ldots, E_{2n-1}$, numbered in such a way that E is the image of E_0, E' is the image of E_n, and E_i meets $E_{i\pm 1}$, with subscripts read modulo $2n$. The elliptic fibration induces an elliptic fibration on R such that $\cup E_i$ is a fibre of Kodaira-type I_{2n}. Near this fibre, the n-torsion curve of R breaks up into a group of sections isomorphic with $\mathbb{Z}_n \times \mathbb{Z}_n$.

The curve $\Delta_n \cap U$ is mapped n-to-one onto a curve $Q_n \subset Q$ and Q_n is transformed birationally into a curve $R_n \subset R$.

$$\begin{array}{ccccc}
\Delta_n & \longrightarrow & Q_n = \cup Q_n^{(j)} & \longleftarrow & R_n = \cup R_n^{(j)} \\
\cap & & \cap & & \cap \\
U & \xrightarrow{q} & Q & \xleftarrow{r} & R \\
\cup & & \cup & & \cup \\
D_k, D'_k & \longrightarrow & E, E' & \longleftarrow & E_0, E_n
\end{array}$$

This curve R_n meets the smooth fibres of R in a group $\mathbb{Z}_n$ of points of order n. As the n-torsion curve on R breaks up into sections, so does R_n. Only the zero-section in R_n will meet E_0. This implies that $R_n = \cup R_n^{(j)}, j = 0, \ldots, n-1$, with the section $R_n^{(j)}$ meeting precisely the curve E_{2j} among all E_j's. The curve $R_n^{(0)}$ maps into Q as a section $Q_n^{(0)}$ through E, which is the image of all the n sections in T_n meeting D_k. If n is even, the curve $R_n^{(n/2)}$ similarly maps into Q as a section meeting E'. The sections $R_n^{(j)}$ for $0 < 2j < n$ map into Q as curves through one singularity, and for $2j > n$ to curves through the other singularity.

Without restriction we now assume $0 < 2j < n$ and denote by $\Delta_n^{(j)} \subset U$ the pullback $q^{-1}(Q_n^{(j)})$ to U of the curve $R_n^{(j)}$.

As the singularities on Q are rational, on a sufficiently small neighbourhood $V \subset R$ of $\bigcup_{j=1}^{n-1} E_j$ the divisors of meromorphic functions are precisely those divisors which have intersection number 0 with each $E_j, j = 1, \ldots, n-1$, cf. [BPV, III.3]. These functions descend to functions on Q and pull back to meromorphic functions on U.

Such principal divisors on R are in particular

$$G^{(j)} = R_n^{(j)} - \sum_{k=0}^{2j-1} (2j-k)\, E_k, \quad G'^{(j)} = R_n^{(j)} - \sum_{k=2j+1}^{n} (k-2j) E_k$$

and there are meromorphic functions $g^{(j)}, g'^{(j)}$ on V with these divisors. They descend to Q and pull back to functions $h^{(j)}, h'^{(j)}$ on U, both meromorphic near one of the two singularities of F_k. Locally their divisors are

$$(h^{(j)}) = \Delta_n^{(j)} - 2j \cdot D_k, \quad (h'^{(j)}) = \Delta_n^{(j)} - (n-2j) \cdot D'_k .$$

This shows that the local intersection numbers are

$$(\Delta_n^{(j)} \cdot D_k) = (n-2j)(D'_k \cdot D_k) = n - 2j, \quad (\Delta_n^{(j)} \cdot D'_k) = 2j (D_k \cdot D'_k) = 2j .$$

Near this point $D_k \cap D'_k$ we now linearize the action of $\mathbb{Z}_n$. I.e., we choose local coordinates x, y such that locally

$$D_k: y = 0, \quad D'_k: x = 0$$

and such that the group action is

$$(x, y) \mapsto (\omega x, \omega^{-1} y), \quad \omega = e^{2\pi i/n}.$$

Then $h^{(j)}$ is invariant under this action. The function $f^{(j)} := y^{2j} \cdot h^{(j)}$ is a local holomorphic equation for $\Delta^{(j)}$ satisfying

$$f(\omega x, \omega^{-1} y) = \omega^{-2j} f(x, y).$$

Its Taylor expansion

$$f(x, y) = \sum_{m, m' = 1}^{\infty} a_{m, m'} x^m y^{m'}$$

therefore contains nonzero coefficients $a_{m,m'}$ only for

$$m - m' \equiv -2j \pmod{n}.$$

The monomials of lowest order possible are

$$x^{n-2j} \text{ and } y^{2j}.$$

Since we know the intersection numbers $(\Delta_n^{(j)} \cdot D_k) = n - 2j$ and $(\Delta_n^{(j)} \cdot D_k') = 2j$, both these monomials in the Taylor expansion have nonzero coefficients. So the multiplicity of $\Delta^{(j)}$ at this point $D_k \cap D_k'$ is $\min\{2j, n - 2j\}$. □

Adding over all components $\Delta_n^{(j)}$, $0 < 2j < n$, we find the intersection multiplicities of Δ_n with D_k and D_k' in each of the two double points p of F_k: For odd n

$$(\Delta_n \cdot D_k)_p = 1 + 3 + \cdots + (n-2) = \frac{(n-1)^2}{4},$$

$$(\Delta_n \cdot D_k')_p = 2 + 4 + \cdots + (n-1) = \frac{n^2 - 1}{4}$$

and for even n

$$(\Delta_n \cdot D_k)_p = (\Delta_n \cdot D_k')_p = 2 + 4 + \cdots + (n-2) = \frac{n^2 - 2n}{4}.$$

Together with the intersection multiplicities in the smooth points of F_i one obtains the intersection numbers:

$$(\Delta_n \cdot D_k) = 2\frac{(n-1)^2}{4} + n - 1 = \frac{1}{2}(n^2 - 1),$$

$$(\Delta_n \cdot D_k') = 2\frac{n^2 - 1}{4} = \frac{1}{2}(n^2 - 1) \quad (n \text{ odd})$$

and

$$(\Delta_n \cdot D_k) = (\Delta_n \cdot D_k') = 2\frac{n^2 - 2n}{4} + n - 2 = \frac{1}{2}(n^2 - 4) \quad (n \text{ even}).$$

3 The modular curve $X_{00}(n, 2)$

The following notations for modular curves seem to be standard [DR, p. 221]:

Modular curve	Parametrizing elliptic curves	Group	$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ s.th.
$X(n)$	With level-n structure	$\Gamma(n)$	$a \equiv d \equiv 1, b \equiv c \equiv 0(n)$
$X_0(n)$	With cyclic subgroup of order n	$\Gamma_0(n)$	$c \equiv 0(n)$
$X_{00}(n)$	With proper n-torsion point	$\Gamma_{00}(n)$	$a \equiv d \equiv 1, c \equiv 0(n)$

Here X is the compactification of $\mathscr{H}/\Gamma$.

For $n > 2$ we are interested in the *modular curve parametrizing elliptic curves with*

- a primitive n-torsion element and
- a level-2 structure.

This curve is the fibre product of $X_{00}(n)$ and $X(2)$ with respect to the j-function map onto the projective line. We call it

$$X_{00}(n, 2) := X_{00}(n) \times_{\mathbb{P}_1} X(2) .$$

(3.1) *If n is odd, then $X_{00}(n, 2)$ is connected. If n is even, then $X_{00}(n, 2)$ decomposes into three connected components, which are isomorphic double covers of $X_{00}(n)$.*

Proof. (a) n odd: A level-n structure on an elliptic curve E is a symplectic isomorphism of $\mathbb{Z}_n \times \mathbb{Z}_n$ onto the n-torsion subgroup of E. The image $e \in E$ of $(1, 0)$ under this isomorphism is a primitive element of order n. Each primitive n-torsion point $e \in E$ belongs in this way to (n different) level structures. Associating the level structure with e defines a surjective map $X(n) \to X_{00}(n)$. This induces a surjective map of $X(n) \times_{\mathbb{P}_1} X(2)$ onto $X_{00}(n, 2)$, and it suffices to show that the fibre product $X(n) \times_{\mathbb{P}_1} X(2)$ is connected.

For odd n the $2n$-torsion subgroup E^{2n} of E is a direct product of E^n and E^2, in a way compatible with the symplectic forms on these groups. So a level-$2n$ structure on E is exactly the same as a level-n structure plus a level-2 structure. This shows that there is an isomorphism of $X(n) \times_{\mathbb{P}_1} X(2)$ with the connected modular curve $X(2n)$.

(b) n even: The choice of an element τ of order n distinguishes one element $\tau^{n/2}$ of order 2. Sending $\tau \mapsto \tau^{n/2}$ defines a map $X_{00}(n) \to X_{00}(2)$. This induces a surjective map

$$X_{00}(n, 2) = X_{00}(n) \times_{\mathbb{P}_1} X(2) \to X_{00}(2) \times_{\mathbb{P}_1} X(2) .$$

There are three different maps over $\mathbb{P}_1$ of $X(2)$ onto $X_{00}(2)$. The fibre product $X_{00}(2) \times_{\mathbb{P}_1} X(2)$ therefore decomposes into three copies of $X(2)$. Then our curve $X_{00}(n, 2)$ decomposes into three curves, which are isomorphic double covers

$$X_{00}(n) \times_{X_{00}(2)} X(2)$$

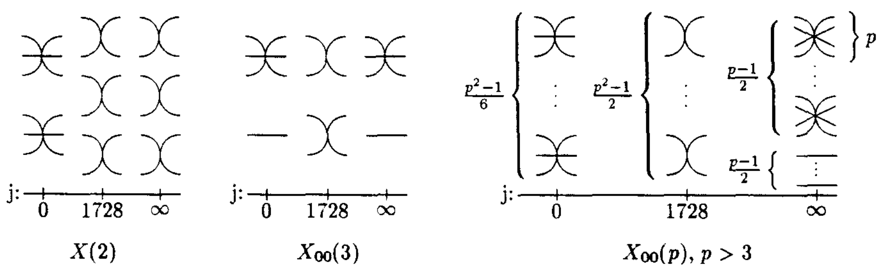
of $X_{00}(n)$.

Multiplication by $\frac{n}{2}$ defines a canonical map $X(n) \to X(2)$. Combining it with $X(n) \to X_{00}(n)$ we get a map of $X(n)$ onto a connected component of the fibre product $X_{00}(n) \times_{X_{00}(2)} X(2)$. There are n distinct level-n structures mapping on the same primitive n-torsion point $e \in E$. One half of them defines another level-2 structure on E as the other half. This implies that the connected component mentioned has degree two over $X_{00}(n)$. So the double cover $X_{00}(n) \times_{X_{00}(2)} X(2) \to X_{00}(n)$ is connected. The curve $X_{00}(n)$ therefore consists of three connected components. □

(3.2) *For an odd prime p*

$$\text{genus}(X_{00}(p, 2)) = \tfrac{1}{4}(p-3)^2 .$$

Proof. The maps $j: X(2) \to \mathbb{P}_1$ and $j: X_{00}(p) \to \mathbb{P}_1$ are of degree six and $(p-1)^2/2$. Their branching patterns are:

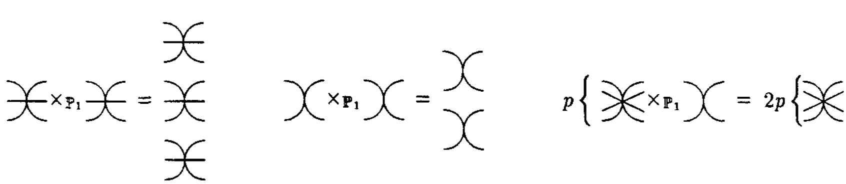


Over the points 0 and 1728 this follows by counting fixed points for the automorphisms of these curves among all primitive n-torsion points. Over ∞ this can be deduced from [BPV, p. 155]: Locally near ∞ there exists a universal family of elliptic curves. The j-function has a simple pole, so the singular fibre here is rational with a node. Near ∞ the period lattice is of the form

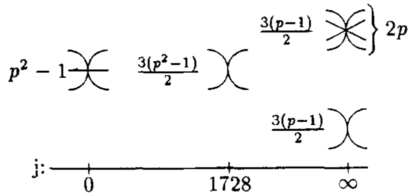
$$\mathbb{Z} \oplus \mathbb{Z} \cdot \frac{1}{2\pi i} \ln(z) .$$

The monodromy on n-torsion points can here be traced easily. We can write the lattice points in a fibre as (a, b) such that this monodromy is $(a, b) \mapsto (a + b, b)$.

Using the rules



we find the branching pattern for $X_{00}(p, 2)$:



Using the genus formula for this branching pattern we compute

$$\begin{aligned}2g-2 &= -2\cdot 3(p^2-1)+(p^2-1)\cdot 2+\tfrac{3}{2}(p^2-1)+\tfrac{3}{2}(p-1)\cdot(2p-1)+\tfrac{3}{2}(p-1)\\ &= -\tfrac{5}{2}(p^2-1)+3\cdot p\cdot(p-1)\\ &= \tfrac{1}{2}\cdot(p^2-6p+5)\\ g &= \tfrac{1}{4}\cdot(p^2-6p+9)\\ &= \tfrac{1}{4}\cdot(p-3)^2\,.\end{aligned}$$

In principle it is not difficult to compute in this way the genus of $X_{00}(n, 2)$ for arbitrary n. The problem is, that the prime decomposition of n makes the branching pattern over ∞ somewhat complicated. So we only consider two examples, $n = 8$ and $n = 12$.

The case $n = 8$. The degree of the covering $X_{00}(8, 2) \to \mathbb{P}_1$ is

$$\tfrac{1}{2}\cdot t(8) = \tfrac{1}{2}\cdot 3\cdot 2^4 = 24\,.$$

Over 0 there lie eight triple points, and twelve double points over 1728. We write the primitive 8-torsion points as pairs $(a, b) \in \mathbb{Z}_8^2$ with $\gcd(a, b, 8) = 1$ such that the monodromy acts by $(a, b) \mapsto (a + b, b)$. Under monodromy they form the following orbits (in brackets):

b	a
0	1, 3, 5, 7
1	(0, 1, 2, 3, 4, 5, 6, 7)
2	(1, 3, 5, 7)
3	(0, 1, 2, 3, 4, 5, 6, 7)
4	(1, 5), (3, 7)

After quotienting out by the involution we obtain the following orbits

Number:	2	1	1	2
Length:	1	2	4	8

and the covering has the following branch points:

j	0	1728	∞		
Number:	48	72	12	6	12
Order:	3	2	2	4	8

This implies for the genus of the total curve

$$\begin{aligned} 2g-2 &= -2\cdot 144+48\cdot 2+72+12+6\cdot 3+12\cdot 7 \\ &= 6\cdot(-48+16+12+2+3+14) \\ &= -6\,. \end{aligned}$$

So the three components of $X_{00}(8, 2)$ are rational.

The case $n = 12$. The degree of the covering $X_{00}(12, 2) \to \mathbb{P}_1$ is

$$\tfrac{1}{2}t(12) = \tfrac{1}{2}(2^2-1)2^2\cdot(3^2-1) = 48\,.$$

Over 0 there lie 16 triple points, and 24 double points over 1728. Writing the primitive 12-torsion points as above the points of order eight, we have the following orbits (in brackets) under monodromy:

b	a
0	1, 5, 7, 11
1	(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
2	(1, 3, 5, 7, 9, 11)
3	(1, 4, 7, 10), (2, 5, 8, 11)
4	(1, 5, 9), (3, 7, 11)
5	(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
6	(1, 7), (5, 11)

As we quotient out by the involution $(a, b) \mapsto (-a, -b)$, there remain the following orbits

Number:	2	1	2	2	1	2
Length:	1	2	3	4	6	12

So the covering has the following branch points:

j	0	1728	∞			
Number:	96	144	12	12	12	12
Order:	3	2	2	4	6	12

and the genus for the curve of three components satisfies

$$\begin{aligned} 2g-2 &= -2\cdot 288+96\cdot 2+144+12\cdot(1+3+5+11) \\ &= 12\cdot(-48+16+12+20) \\ &= 0 \end{aligned}$$

The three components of $X_{00}(12, 2)$ are elliptic.

Let us denote by $\Delta'_n \subset \Delta_n$ the closure of the set of those points, which on their fibre are primitive n-torsion points. Then the universal property of the modular curve $X_{00}(n, 2)$ defines a morphism

$$\text{normalization}(\Delta'_n) \to X_{00}(n, 2)\,.$$

(3.3) *This morphism is a double cover, identifying each point in* Δ'_n *with is inverse* (w.r.t. *the addition on* $F_{(\lambda:\mu)}$, *if the point lies on the smooth fibre* $F_{(\lambda:\mu)}$.)

Proof. It suffices to prove the assertion outside of the singular fibres $F_{(\lambda:\mu)}$. Points in Δ'_n on different fibres have different images in $X_{00}(n, 2)$, because if two different fibres are isomorphic, they will differ by their level structure, cf. (1.1). Two points on the same fibre $F_{(\lambda:\mu)}$ with the same image in $X_{00}(n, 2)$ are equivalent under an automorphism of the elliptic curve $F_{(\lambda:\mu)}$. In general this automorphism can only be $\pm$ identity. □

4 The plane curve Π_n

We define the plane curve

$$\Pi_n := (\gamma\tau_2\tau_1)\Delta_n \subset \mathbb{P}_2 .$$

On each smooth fibre $F_{(\lambda:\mu)}$ the restriction of $\gamma\tau_2\tau_1$ is the quotient map with respect to the (-1)-involution. This shows that $\Delta_n \to \Pi_n$ is a double cover. In fact, on the part of Δ'_n belonging to smooth fibres this is the map from (3.3). So, if we put

$$\Pi'_n := (\gamma\tau_2\tau_1)\Delta'_n \subset \Pi_n ,$$

then:

(4.1) *The plane curve Π'_n is a birational image of the modular curve $X_{00}(n, 2)$.*

There is a formula for the *equation* g_n of Π_n, cf. Sect. 5. In practice this formula seems however too complicated to be evaluated by hand. We want to compute the equation in the first few cases in another way. So we collect first a few simple properties of Π_n. They determine its equation uniquely for low n and make it easier to calculate g_n.

In (2.1) we observed that Δ_n does not meet the curves $C_i, i = 0, \ldots, 3$. This implies that Π_n does not pass through any base point $P_0, \ldots, P_3$. So Π_n meets the smooth conics $C_{(\lambda:\mu)}$ in $\frac{n^2-1}{2}$ points, if n is odd, and in $\frac{n^2}{2} - 2$ points, if n is even. This gives the degree

$$\deg(\Pi_n) = \begin{cases} (n^2-1)/4 & \text{if } n \text{ is odd} \\ n^2/4 - 1 & \text{if } n \text{ is even} . \end{cases}$$

Let us denote by $\Sigma_4 \subset \mathrm{PGL}(2, \mathbb{C})$ the symmetric group generated by permutations of the four base points $P_k = (\pm 1 : \pm 1 : \pm 1)$. It contains the subgroup

$$\mathbb{Z}_2 \times \mathbb{Z}_2 : (x_0 : x_1 : x_2) \mapsto (\pm x_0 : \pm x_2 : \pm x_2) .$$

The fix-group of P_0 is a copy Σ_3 of the symmetric group permuting the three base points P_1, P_2, P_3, or, the three coordinates.

We saw already (Sect. 2) that the map $X \to \mathbb{P}_2$ is equivariant for $\mathbb{Z}_2 \times \mathbb{Z}_2$. Unfortunately the Σ_3-symmetries do not lift to X, because they would move the singular fibre of type I_0^*

(4.2) a) *For all $n > 2$ the polynomial g_n is symmetric in the three coordinates.*

b) *If $n > 2$ is even, the curve Π_n is invariant under Σ_4. Its equation g_n then is a symmetric polynomial in the squares x_0^2, x_1^2, and x_2^2, or such a polynomial times $x_0 x_1 x_2$. (Of course in this case g_n has odd degree.)*

Proof. We observed already that the elliptic curve $F_{(\lambda:\mu)}$ depends only on the position of the base points on the conic $C_{(\lambda:\mu)}$, not on the particular surface X, i.e. not on the choice of the branch conic C. The same of course holds for the intersection of Π_n with $C_{(\lambda:\mu)}$. Any $\phi \in \Sigma_3$ transports the images of torsion points over $C_{(\lambda:\mu)}$ to those on $\phi(C_{(\lambda:\mu)})$. This implies $\phi(\Pi_n) = \Pi_n$. Each S_3-symmetry therefore multiplies the polynomial g_n by ± 1. The multiplier -1 is impossible, because, this would imply that Π_n would pass through the base point $P_0 = (1:1:1)$.

If n is even, the n-torsion subgroup on the elliptic curve is unchanged when the origin is replaced by a nontrivial element of order two. Addition by one of the three nontrivial two-torsion sections S_1, S_2, or S_3 induces on $\mathbb{P}_2$ an involution $(x_0:x_1:x_2) \mapsto (x_0: \pm x_1: \pm x_2)$. This symmetry therefore multiplies the equation g_n by ± 1. If we have $+1$ here, g_n is a polynomial in the squares x_0^2, x_1^2, x_2^2. If we have -1, the polynomial g_n is a sum of expressions

$$x_0^a x_1^b x_2^c + x_0^b x_1^c x_2^a + x_0^c x_1^a x_2^b + x_0^a x_1^c x_2^b + x_0^c x_1^b x_2^a + x_0^b x_1^a x_2^c$$

with odd exponents a, b, c. □

The four-torsion curve Π_4 is easy to find: The four-torsion points on $F_{(\lambda:\mu)}$ having as square the intersection points $F_{(\lambda:\mu)} \cap S_m$, $m = 1, 2, 3$, are exactly those points, which under addition with S_m go to their inverses. Their images in $\mathbb{P}_2$ are the fixed points for the involution belonging to S_m, i.e. a coordinate line. This already describes

$$\Pi_4 : x_0 x_1 x_2 = 0 .$$

The singular fibres $F_k = D_k + D'_k$ are mapped onto line pairs $L_k + L'_k$. Their equations and intersections are

k	L_k	L'_k	Intersection
0	$x_1 = x_2$	$x_1 = -x_2$	$(1:0:0)$
1	$x_0 = x_2$	$x_0 = -x_2$	$(0:1:0)$
2	$x_0 = x_1$	$x_0 = -x_1$	$(0:0:1)$

The two double points of the fibre F_k both are mapped onto the intersection point of L_k and L'_k. Since the map $\gamma\tau_2\tau_1$ is etale near these points, the local branches $\Delta_n^{(j)}$ map isomorphically onto (not necessarily irreducible) branches $\Pi_n^{(j)}$ of Π_n having the same multiplicities as $\Delta_n^{(j)}$, and the same intersection numbers with L_k and L'_k as $\Delta_n^{(j)}$ with D_k and D'_k:

n	Degree of Π_n	Branches	Intersection with L_k, L'_k	Multiplicity
Odd	$(n^2-1)/4$	$(n-1)/2$	$(1, n-1), (3, n-3), \ldots, (n-2, 2)$	$(n^2-1)/8$
$\equiv 0 \mod 4$	$n^2/4 - 1$	$n/2 - 1$	$(2, n-2), (4, n-4), \ldots, (n-2, 2)$	$n^2/8$
$\equiv 2 \mod 4$	"	"	"	$(n^2-4)/8$

A branch $\Delta_n^{(j)}$ with intersection numbers $2j, n-2j$ by 2.2 has a local equation $a \cdot x^{2j} + b \cdot y^{n-2j} + \cdots = 0$, so it is tangent to one of the curves D_k, D'_k unless $n = 2j$.

Since the singularities of Π_n are concentrated in the coordinate points, the curve simplifies under the standard Cremona transform

$$(x_0:x_1:x_2) \mapsto (x_1 \cdot x_2 : x_2 \cdot x_0 : x_0 \cdot x_1)$$

$$\Pi_n \to \mathrm{Crem}(\Pi_n) .$$

This Cremona transformation is equivariant for the symmetry group Σ_4, so $\mathrm{Crem}(\Pi_n)$ has the same symmetry properties as the curve Π_n.

One easily computes

$$\mathrm{degree}(\mathrm{Crem}(\Pi_n)) = 2 \cdot \mathrm{degree}(\Pi_n) - \text{multiplicities in the coordinate points}$$

and finds

n	Odd	$\equiv 0 \bmod 4$	$\equiv 2 \bmod 4$
$\deg(\mathrm{Crem}(\Pi_n))$	$\frac{n^2-1}{8}$	$\frac{n^2}{8} - 2$	$\frac{n^2}{8} - \frac{1}{2}$

(Of course, this holds only if Π_n does not contain coordinate lines, i.e. for $n \neq 4$.)

For the computations we use the following symmetric polynomials

$$s_1 = x_0 + x_1 + x_2$$

$$s_2 = x_0x_1 + x_0x_2 + x_1x_2$$

$$s_3 = x_0x_1x_2 .$$

Then the equation for $\mathrm{Crem}(\Pi_n)$ is a symmetric polynomial $f_n(s_1, s_2, s_3)$ of total degree d. We collect some of its properties:

(a) *Intersection with L'_k, n odd.* The line L'_k is invariant under the Cremona transformation. The coordinate point $L_k \cap L'_k$ is blown up and mapped to the opposite coordinate line $x_k = 0$. The restriction of g_n to L'_k vanishes only at $L_k \cap L'_k$, so $f_n | L'_k$ vanishes only at the intersection of L'_k with the coordinate line, which coincides with the intersection of L'_k with the line $s_1 = 0$. Since the union $L_0 \cup L_1 \cup L_2$ has equation

$$(x_0 + x_1)(x_0 + x_2)(x_1 + x_2) = s_1 \cdot s_2 - s_3 ,$$

we find

$$f_n(s_1, s_2, s_3) = s_1^d + (s_1 s_2 - s_3) \cdot p_{d-3}(s_1, s_2, s_3) .$$

Here $d = \deg(\mathrm{Crem}(\Pi_n))$ and p_{d-3} is some symmetric polynomial of degree $d - 3$.

(b) *Intersection with L_k.* Just as L'_k, the line L_k too is invariant under the Cremona transformation. The intersection points of Π_n with L_k outside of the coordinate point $L_k \cap L'_k$ are the images of the $n - 1$ (resp. $n - 2$, if n is even) nontrivial n-torsion points. So $g_n | L_k$ has $(n-1)/2$ $\left(\text{resp. } \frac{n-2}{2} \text{ if } n \text{ is even}\right)$ zeros outside of $L_k \cap L'_k$, and $f_n | L_k$ has the same number of zeros outside of the coordinate line $z_k = 0$. The polynomial h_n of degree $(n-1)/2$, resp. $\frac{n-2}{2}$, vanishing in these points can – in principle – be computed:

Consider the line $L_0 : x_1 = x_2$. Introduce homogeneous coordinates $u : v$ on D_0 such that $u \cdot v$ vanishes on the two intersections of D_0 with D'_0, and such that $(u:v) = (1:1)$ is the origin $D_0 \cap S_0$. The map $D_0 \to L_0$ is given by

$$x_0 = u^2 + v^2, \quad x_1 = x_2 = 2 \cdot uv .$$

The nontrivial n-torsion points on D_0 are the roots of

$$\frac{u^n - v^n}{u - v} = u^{n-1} + u^{n-1}v + \cdots + v^{n-1}\,.$$

Writing this as polynomial in x_0 and x_1 gives *for odd n* e.g:

$$h_3 = x_0 + \tfrac{1}{2}x_1$$

$$h_5 = x_0^2 + \tfrac{1}{2}x_0x_1 - \tfrac{1}{4}x_1^2$$

$$h_7 = x_0^3 + \tfrac{1}{2}x_0^2x_1 - \tfrac{1}{2}x_0x_1^2 - \tfrac{1}{8}x_1^3$$

$$h_9 = x_0^4 + \tfrac{1}{2}x_0^3x_1 - \tfrac{3}{4}x_0^2x_1^2 - \tfrac{1}{4}x_0^3x_1 + \tfrac{1}{16}x_1^4\,.$$

For *even n* the polynomial $f_n \mid L_0$ has $\frac{n}{2} - 1$ zeros outside $(0:1:1)$, the images of the $n - 2$ points $\neq \pm 1$ of order n on D_0. The equation for these zeros is obtained by expanding

$$\frac{u^n - v^n}{u^2 - v^2} = (u^2)^{n/2-1} + (u^2)^{n/2-3}v^2 + \cdots + (v^2)^{n/2-1}$$

in x_0 and x_1:

$$h_6 = x_0^2 - \tfrac{1}{4}x_1^2$$

$$h_8 = x_0^3 - \tfrac{1}{2}x_0x_1^2$$

$$h_{10} = x_0^4 - \tfrac{3}{4}x_0^2x_1^2 + \tfrac{1}{16}x_1^4$$

$$h_{12} = x_0^5 - x_0^3x_1^2 + \tfrac{3}{16}x_0x_1^4\,.$$

This polynomial $h_n(x_0, x_1)$ vanishes on the intersection of L_0 with Π_n outside of $(1:0:0)$ and – after the Cremona transform – $h_n(x_1, x_0)$ vanishes on the intersection of L_0 with $\mathrm{Crem}(\Pi_n)$ outside of the line $x_0 = 0$.

(c) *Intersection with coordinate lines.* A branch $\Pi_n^{(j)}$ of Π_n at $(1:0:0)$ with intersection numbers $(\Pi_n^{(j)} \cdot L_0) = n - 2j$, $(\Pi_n^{(j)} \cdot L_0') = 2j$ goes under the Cremona transform to a branch with intersection numbers

L_0	L_0'	$x_0 = 0$	
$n - 4j$	0	$2j$	$(4j < n)$
0	$4j - n$	$n - 2j$	$(4j > n)$

So $\mathrm{Crem}(\Pi_n)$ meets the coordinate line $x_0 = 0$ at its intersection $(0:1:1)$ with L_0 with multiplicity

$$\sum_{1 \le j < \frac{n}{4}} 2j = k(k+1) = \left\{\begin{matrix} (n-1)(n+3)/16 \\ (n^2-4)/16 \\ (n+1)(n-3)/16 \\ n(n-4)/16 \end{matrix}\right\} \quad \text{if } n = 4k + \left\{\begin{matrix} 1 \\ 2 \\ 3 \\ 4\,. \end{matrix}\right.$$

And at the intersection $(0:1:-1)$ with the line L'_0 the multiplicity is

$$n = 4k: \quad \sum_{k<j<\frac{n}{2}} (n-2j) = \sum_{j=1}^{k-1} (2k-2j) = k(k-3) \quad = n(n-4)/16$$

$$n = 4k+1: \quad \sum_{k<j<\frac{n}{2}} (n-2j) = \sum_{j=1}^{k} (2k+1-2j) = k^2 \quad = (n-1)^2/16$$

$$n = 2k+2: \quad \sum_{k<j<\frac{n}{2}} (n-2j) = \sum_{j=1}^{k} (2k+2-2j) = k(k+1) \quad = (n^2-4)/16$$

$$n = 4k+3: \quad \sum_{k<j<\frac{n}{2}} (n-2j) = \sum_{j=1}^{k+1} (2k+3-2j) = (k+1)^2 \quad = (n+1)^2/16 .$$

For odd n the two intersection numbers with the coordinate line add up to $\deg(\mathrm{Crem}(\Pi_n)) = (n^2-1)/8$. Since the three intersections of the lines L'_k with the coordinate lines $x_k = 0$ are on the line $s_1 = 0$, and since the conic $s_1^2 - 4s_2 = 0$ touches the coordinate lines at their intersection with L_k, the curve $\mathrm{Crem}(\Pi_n)$ intersects the coordinate triangle $s_3 = 0$ as the curve $s_1^{m_1} \cdot (s_1^2 - 4s_2)^{m_2} = 0$, with m_1 and m_2 the intersection multiplicities of $\mathrm{Crem}(\Pi_n)$ with the coordinate lines just computed. We conclude

$$f_n = s_3 \cdot q_{d-3} + \begin{Bmatrix} s_1^{(n-1)^2/16} \cdot (s_1^2 - 4s_2)^{(n-1)(n+3)/32} \\ s_1^{(n+1)^2/16} \cdot (s_1^2 - 4s_2)^{(n+1)(n-3)/32} \end{Bmatrix} \quad \text{if} \quad \begin{cases} n \equiv 1(4) \\ n \equiv 3(4) \end{cases} .$$

(d) *Intersection with the line* $s_1 = 0$ $(n > 3)$. The first equation we determine below will be $f_3 = s_1$. The curves $\mathrm{Crem}(\Pi_n)$, $n > 3$ do not meet $\mathrm{Crem}(\Pi_3)$ outside of the coordinate lines. By Σ_3-symmetry, the intersection multiplicities at the three points of intersection are equal. Hence $\mathrm{Crem}(\Pi_n)$ cuts out the same divisor on the line $s_1 = 0$ as the polynomial $s_3^{d/3}$, where $d := \deg(\mathrm{Crem}(\Pi_n))$. This implies

$$f_n = s_3^{d/3} + s_1 r_{d-1}(x_0, x_1, x_2) .$$

Now we determine the polynomial f_n in the few cases. For polynomials depending only on the squares of the coordinates x_0, x_1, x_2 we abbreviate

$$\sigma_1 := x_0^2 + x_1^2 + x_2^2, \qquad \sigma_2 := x_0^2 x_1^2 + x_0^2 x_2^2 + x_1^2 x_2^2, \qquad \sigma_3 := x_0^2 x_1^2 x_2^2 .$$

The case $n = 3$. The symmetric polynomial f_3 is linear, so $f_3 = s_1$.

The case $n = 5$. The symmetric polynomial f_5 has degree three. We have

$$\begin{aligned} f_5 &= s_1^3 + a(s_1 s_2 - s_3),\ a \in \mathbb{C} , \qquad &(\mathrm{a}) \\ &= c s_3 + s_1(s_1^2 - 4s_2),\ c \in \mathbb{C} , \qquad &(\mathrm{c}) . \end{aligned}$$

This implies $a = -c = -4$. So

$$\begin{aligned} f_5 &= s_1^3 - 4s_1 s_2 + 4s_3 \\ &= x_0^3 + x_1^3 + x_2^3 - (x_0^2 x_1 + \cdots + x_1 x_2^2) - 2x_0 x_1 x_2 . \end{aligned}$$

This is the equation of a smooth cubic. The curve $X_{00}(5, 2)$ has *genus one* (cf. 4.2).

The case $n = 6$. The curve $\mathrm{Crem}(\Pi_6)$ has degree four, it contains $\mathrm{Crem}(\Pi_3)$, and it is invariant under sign-changes of the coordinates. This implies

$$\begin{aligned} f_6 &= (x_0 + x_1 + x_2)(x_0 + x_1 - x_2)(x_0 - x_1 + x_2)(-x_0 + x_1 + x_2) \\ &= x_0^4 + x_1^4 + x_2^4 - 2(x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2) \\ &= \sigma_1^2 - 4\sigma_2 \,. \end{aligned}$$

After dividing by $-s_1$, the equation for $\mathrm{Crem}(\Pi_3)$, we find

$$\begin{aligned} f_6' &= -(s_1 - 2x_0)(s_1 - 2x_1)(s_1 - 2x_2) \\ &= s_1^3 - 4s_1s_2 + 8s_3 \,. \end{aligned}$$

The case $n = 7$. The symmetric polynomial f_7 has degree six. We know

$$\begin{aligned} f_7 &= s_1^6 + (s_1s_2 - s_3)\cdot p_3 \qquad &\text{(a)} \\ &= s_3\cdot q_3 + s_1^4(s_1^2 - 4s_2) \qquad &\text{(c)} \,. \end{aligned}$$

This implies $p_3 = -4s_1^3 + cs_3,\ c \in \mathbb{C}$. Thus

$$\begin{aligned} f_7(1 + 2x, 2x + x^2, x^2) &= (1 + 2x)^6 + (2x + 4x^2 + 2x^3)(-4(1 + 2x)^3 + cx^2) \\ &= (1 + 2x)^6 + 2x\cdot\{-4 - 32x + (-100 + c)x^2 \\ &\quad + (-152 + 2c)x^3 + (-112 + c)x^4 - 32x^5)\} \end{aligned}$$

is a polynomial of degree three, hence $c = 112 - 6\cdot 2^4 = 16$. We computed

$$\begin{aligned} f_7 &= s_1^6 + (s_1s_2 - s_3)(-4s_1^3 + 16s_3) \\ &= s_1^6 - 4s_1^4s_2 + 4s_1^3s_3 + 16s_1s_2s_3 - 16s_3^2 \\ &= x_0^6 + x_1^6 + x_2^6 + 2(x_0^5x_1 + \cdots + x_1x_2^5) - (x_0^4x_1^2 + \cdots + x_1^2x_2^4) \\ &\quad - 4(x_0^3x_1^3 + x_0^3x_2^3 + x_1^3x_2^3) - 2x_0x_1x_2(x_0^3 + x_1^3 + x_2^3) + 2x_0^2x_1^2x_2^2 \,. \end{aligned}$$

One checks that the curve $\mathrm{Crem}(\Pi_7)$ has two double points on each coordinate line. E.g. on the line $x_0 = 0$ these are $(0:1:1)$ and $(0:1:-1)$. In the latter point one branch of the double point even has threefold contact with the coordinate line.

The case $n = 8$. $\mathrm{Crem}(\Pi_8)$ is a curve of degree six. By (4.2), b its equation f_8 is a symmetric polynomial in x_0^2, x_1^2, x_2^2. We put

$$f_8(x_0, x_1, x_2) = a\sigma_1^3 + b\sigma_1\sigma_2 + c\sigma_3 \,.$$

By (b) its restriction to L_0

$$\begin{aligned} f_8(1, x, x) &= a(1 + 2x^2)^3 + b(1 + 2x^2)(2x^2 + x^4) + cx^4 \\ &= (8a + 2b)x^6 + (12a + 5b + c)x^4 + (6a + 2b)x^2 + a \end{aligned}$$

is a polynomial of degree $\leqq 3$, hence

$$a = 1, b = -4, c = 8 \,.$$

We computed

$$\begin{aligned} f_8 &= \sigma_1^3 - 4\sigma_1\sigma_2 + 8\sigma_3 \\ &= (x_0^2 + x_1^2 + x_2^2)^3 - 4(x_0^2 + x_1^2 + x_2^2)(x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2) + 8x_0^2x_1^2x_2^2 \\ &= x_0^6 + x_1^6 + x_2^6 - (x_0^4x_1^2 + \cdots + x_1^2x_2^4) + 2x_0^2x_1^2x_2^2 \\ &= -(x_0^2 + x_1^2 - x_2^2)(x_0^2 - x_1^2 + x_2^2)(-x_0^2 + x_1^2 + x_2^2) \,. \end{aligned}$$

The case $n = 9$. The polynomial f_9 has degree ten and splits off the linear factor $f_3 = s_1$. We write $f_9 = s_1 f_9'$ with

$$f_9' = s_1^9 + (s_1 s_2 - s_3)(as_1^6 + bs_1^4 s_2 + cs_1^3 s_3 + ds_1^2 s_2^2 + es_1 s_2 s_3 + fs_2^3 + gs_3^2) \qquad \text{(a)}$$
$$= s_1^3(s_1^2 - 4s_2)^3 + s_3(a's_2^6 + b's_1^4 s_2 + c's_1^3 s_3 + d's_1^2 s_2^2$$
$$+ e's_1 s_2 s_3 + f's_2^3 + g's_3^2) \qquad \text{(c)} .$$

This implies $a = -12$, $b = 48$, $d = -64$, and $f = 0$, so

$$f_9' = s_1^9 + s_1^2(s_1 s_2 - s_3)(-12s_1^4 + 48s_1^2 s_2 + 64s_2^2)$$
$$+ (s_1 s_2 - s_3)s_3(cs_1^3 + es_1 s_2 + gs_3) .$$

This polynomial restricts to L_k as

$$f_9'|L_k = (1 + 2x)^9 + (1 + 2x)^2(2x + 4x^2 + 2x^3)$$
$$\cdot\{-12(1 + 2x)^4 + 48(1 + 2x)^2(2x + x^2) - 64(2x + x^2)^2)\}$$
$$+ (2x + 4x^2 + 2x^3)x^2 \cdot \{c(1 + 2x)^3 + e(1 + 2x)(2x + x^2) + gx^2\}$$
$$= 1 - 6x + (136 + 2c)x^3 + (256 + 16c + 4e)x^4$$
$$+ (128 + 50c + 18e + 2g)x^5 + (256 + 76c + 28e + 4g)x^6$$
$$+ (512 + 56c + 18e + 2g)x^7 + (256 + 16c + 4e)x^8 .$$

All terms of degree $\geqq 4$ should vanish, hence $c = -64$, $e = 192$, and $g = -192$. We computed

$$f_9' = s_1^9 + (s_1 s_2 - s_3)(-12s_1^6 + 48s_1^4 s_2 - 64s_1^3 s_3 - 64s_1^2 s_2^2 + 192s_1 s_2 s_3 - 192s_3^2)$$
$$= s_1^9 - 3\cdot 2^2 s_1^7 s_2 + 3\cdot 2^2 s_1^6 s_3 + 3\cdot 2^4 s_1^5 s_2^2 - 7\cdot 2^4 s_1^4 s_2 s_3$$
$$+ 2^6 s_1^3 s_3^2 - 2^6 s_1^3 s_2^3 + 2^8 s_1^2 s_2^2 s_3 - 3\cdot 2^7 s_1 s_2 s_3^2 + 3\cdot 2^8 s_3^3 .$$

The case $n = 10$. The curve Crem(Π_{10}) has degree twelve, it contains Crem(Π_5), and it is invariant under sign-changes of the coordinates. This implies

$$f_{10} = (x_0^3 + x_1^3 + x_2^3 - (x_0^2 x_1 + x_0^2 x_2 + x_0 x_1^2 + x_1^2 x_2 + x_0 x_2^2 + x_1 x_2^2) - 2x_0 x_1 x_2) .$$
$$(x_0^3 + x_1^3 - x_2^3 - (x_0^2 x_1 - x_0^2 x_2 + x_0 x_1^2 - x_1^2 x_2 + x_0 x_2^2 + x_1 x_2^2) + 2x_0 x_1 x_2) .$$
$$(x_0^3 - x_1^3 + x_2^3 - (-x_0^2 x_1 + x_0^2 x_2 + x_0 x_1^2 + x_1^2 x_2 + x_0 x_2^2 - x_1 x_2^2) + 2x_0 x_1 x_2) .$$
$$(-x_0^3 + x_1^3 + x_2^3 - (x_0^2 x_1 + x_0^2 x_2 - x_0 x_1^2 + x_1^2 x_2 - x_0 x_2^2 + x_1 x_2^2) + 2x_0 x_1 x_2) .$$

or

$$f_{10}' = \prod_{k=0}^{2} (s_1^3 - 4s_1 s_2 + 8s_3 + x_k(s_1^2 - 2s_2) - 2x_k^2) .$$

The case $n = 11$. We did not manage to compute f_{11}. It seems, the properties collected so far do not determine the curve Crem(Π_{11}) uniquely.

The case $n = 12$. The polynomial f_{12} is of degree 16. Fortunately it splits off f_6 as a factor. We write f_{12}' for the polynomial f_{12}/f_6. This polynomial of degree twelve is symmetric in x_0^2, x_1^2, x_2^2. So

$$f_{12}' = a\sigma_1^6 + b\sigma_1^4\sigma_2 + c\sigma_1^3\sigma_3 + d\sigma_1^2\sigma_2^2 + e\sigma_1\sigma_2\sigma_3 + f\sigma_2^3 + g\sigma_3^2 .$$

By (b) this polynomial restricts to L_0 as

$$\frac{h_{12}(x,1)}{h_6(x,1)} = \frac{16x^5 - 16x^3 + 3x}{4x^3 - x} = 4x^2 - 3 .$$

Evaluating the linear conditions imposed by this equation we find

$$a = -3, b = 20, c = -(160 + 4d), e = 672 + 18d,$$
$$f = -(128 + 4d), g = -(944 + 27d) .$$

Unfortunately there is the free parameter d not determined yet. We restrict this equation to the line $x_0 = 0$:

$$-3(x_1^2 + x_2^2)^6 + 20(x_1^2 + x_2^2)^4 x_1^2 x_2^2 + d(x_1^2 x_2^2)^2 x_1^4 x_2^4 - (128 + 4d) x_1^6 x_2^6$$
$$= ((x_1^2 + x_2^2)^2 - 4x_1^2 x_2^2)(-3(x_1^2 + x_2^2)^4 + 8(x_1^2 + x_2^2)^2 x_1^2 x_2^2 + (d + 32) x_1^4 x_2^4 .$$

By (c) this polynomial should split off the square of

$$(x_1^2 + x_2^2)^2 - 4x_1^2 x_2^2 = (x_1^2 - x_2^2)^2 .$$

This now determines $d = -16$ and we have

$$f_{12}' = -3\sigma_1^6 + 5\cdot 2^2 \sigma_1^4 \sigma_2 - 3\cdot 2^5 \sigma_1^3 \sigma_3 - 2^4 \sigma_1^2 \sigma_2^2 + 3\cdot 2^7 \sigma_1 \sigma_2 \sigma_3 - 2^6 \sigma_2^3 - 2^9 \sigma_3^2 .$$

From Sect. 3 we know that this curve splits into three elliptic quartics, but we did not try to find their equations.

The curve $\mathrm{Crem}(\Pi_n)$ is Cremona-transformed back to Π_n by the substitutions

$$s_1 \mapsto s_2, \qquad s_2 \mapsto s_1 s_3, \qquad s_3 \mapsto s_3^2 .$$

The equations obtained are given in the following table:

$$g_3 = s_2$$

$$g_4 = s_3$$

$$g_5 = -4 s_1 s_2 s_3 + s_2^3 + 4 s_3^2$$

$$g_6 = \sigma_2^2 - 4\sigma_1 \sigma_3$$

$$g_6' = \prod (s_2 - x_k x_{k+1})$$

$$g_7 = -4 s_1 s_2^4 s_3 + 16 s_1 s_2 s_3^3 + s_2^6 + 4 s_2^3 s_3^2 - 16 s_3^4$$

$$g_8 = s_3 \cdot (-4\sigma_1 \sigma_2 \sigma_3 + \sigma_2^3 + 8\sigma_3^2)$$

$$g_8' = (x_0^2 x_1^2 + x_0^2 x_2^2 - x_1^2 x_2^2)(x_0^2 x_1^2 - x_0^2 x_2^2 + x_1^2 x_2^2)(-x_0^2 x_1^2 + x_0^2 x_2^2 + x_1^2 x_2^2)$$

$$g_9' = -2^6 \cdot s_1^3 s_2^3 s_3^3 + 3\cdot 2^4 \cdot s_1^2 s_2^5 s_3^2 - 3\cdot 2^4 \cdot s_1 s_2^7 s_3 - 7\cdot 2^4 \cdot s_1 s_2^4 s_3^3$$
$$+ 2^8 \cdot s_1^2 s_2^2 s_3^4 - 3\cdot 2^7 \cdot s_1 s_2 s_3^5 + s_2^9 + 3\cdot 2^2 \cdot s_2^6 s_3^2 + 2^6 \cdot s_2^3 s_3^4 + 3\cdot 2^8 \cdot s_3^6$$

$$g_{10}' = \prod_k (-4\cdot s_1 s_2 s_3 + s_2^3 + 8 s_3^2 - x_k x_{k+1}(2\cdot s_1 s_3 - s_2^2) - 2\cdot x_k^3 x_{k+1}^3)$$

$$g_{12}' = -2^6 \cdot \sigma_1^3 \sigma_3^3 - 2^4 \cdot \sigma_1^2 \sigma_2^2 \sigma_3^2 + 5\cdot 2^2 \cdot \sigma_1 \sigma_2^4 \sigma_3 + 3\cdot 2^7 \cdot \sigma_1 \sigma_2 \sigma_3^2 - 3\cdot \sigma_2^6$$
$$- 3\cdot 2^5 \cdot \sigma_2^3 \sigma_3^2 - 2^9 \cdot \sigma_3^4 .$$

5 Poncelet polygons

The double cover $E_{(\lambda:\mu)} \to C_{(\lambda:\mu)}$ from Sect. 1, branched over the base points $P_0, \ldots, P_3$, has a geometric interpretation in the plane $\mathbb{P}_2$: We fix a smooth conic $C \neq C_{(\lambda:\mu)}$ in the pencil and consider pairs x, L where $x \in C_{(\lambda:\mu)}$ and L is a tangent line to C through the point x. Since the general point $x \in C_{(\lambda:\mu)}$ admits two tangents to C, the projection $(x, L) \mapsto x$ is a double cover of $C_{(\lambda:\mu)}$. It is branched precisely over the four base points, because at these four intersections $x \in C_{(\lambda:\mu)} \cap C$ the two tangents from x to C coincide.

It is easy to check, e.g. by explicit equations, that these pairs $(x, L) \in C_{(\lambda:\mu)} \times C^* \simeq \mathbb{P}_1 \times \mathbb{P}_1$ form a smooth curve (of bidegree (2, 2)), if both conics $C_{(\lambda:\mu)}$ and C are smooth. So this double cover of $C_{(\lambda:\mu)}$ is an elliptic curve isomorphic with $E_{(\lambda:\mu)}$. Poncelet's theorem is proved easily using this double cover (cf. e.g. [GH]):

With each pair $x, L \in E_{(\lambda:\mu)}$ we can naturally associate another such pair x', L', where x' is the 'second' intersection of L with the conic $C_{(\lambda:\mu)}$ and L' is the 'second' tangent to C through x'. One checks, that this map $(x, L) \mapsto (x', L')$ has no fixed point on the elliptic curve $E_{(\lambda:\mu)}$, hence it is a translation on this curve. (The inverse of this translation is the map $x, L \mapsto x'', L''$ with L'' the second tangent through x and x'' the second intersection of L'' with $C_{(\lambda:\mu)}$.) If there is a Poncelet n-gon circumscribed about C and inscribed into $C_{(\lambda:\mu)}$, then this translation is of order n. And if the translation has order n, then its application to an arbitrary pair $x, L \in E_{(\lambda:\mu)}$ yields a Poncelet n-gon. This proves

Poncelet's theorem. *If there is one n-gon circumscribed about the smooth conic C and inscribed into the smooth conic $C_{(\lambda:\mu)}$, then there is an infinity of such n-gons.*

We endowed the elliptic curve $E_{(\lambda:\mu)}$ already with an origin, the point over P_0. The Poncelet translation described above maps this point to the pair P, T where $T := T_{P_0}(C)$ is the tangent to C at P_0 and P is our *control point*

$$P := T_{P_0}(C) \cap C_{(\lambda:\mu)}$$

on $C_{(\lambda:\mu)}$, the second intersection of this conic with T. Clearly, the Poncelet translation is n-torsion if and only if the point $P, T \in E_{(\lambda:\mu)}$ is n-torsion, i.e., if and only if *the control point P belongs to the curve Π_n*. This is the essential new, however easy observation of our paper.

(5.1) *A smooth conic C in our pencil is n-inscribed into the smooth conic D in the pencil if and only if the control point $P = D \cap T_{P_0}(C)$ belongs to Π_n.*

Of course the meaning of n-torsion is a little ambiguous. If it means primitive n-torsion, i.e., if a Poncelet triangle is not counted as special form of a Poncelet hexagon, then in our statement the curve Π_n should be replaced by $\Pi_n' \subset \Pi_n$, where all curves Π_k, $k|n$, are removed.

Griffiths and Harris [GH] used a formula of Cayley to put the condition that D be n-circumscribed into the form of a symmetric determinant: Take one indeterminate t to write in form of a power series

$$\sqrt{\frac{\det(tD + C)}{\det(C)}} = 1 + A_1 t + A_2 t^2 + \cdots.$$

Then D is n-circumscribed about C if and only if

$$\det\begin{pmatrix} A_2 & \cdots & A_{m+1} \\ \vdots & & \vdots \\ A_{m+1} & \cdots & A_{2m} \end{pmatrix} = 0\,, \qquad (n = 2m+1)$$

$$\det\begin{pmatrix} A_3 & \cdots & A_{m+1} \\ \vdots & & \vdots \\ A_{m+1} & \cdots & A_{2m-1} \end{pmatrix} = 0\,, \qquad (n = 2m)\,.$$

This determinant gives the equation f_n of our curve Π_n in the following way: A point $x = (x_0 : x_1 : x_2) \in \mathbb{P}_2$ lies on Π_n if and only if the conic

$$D := \begin{pmatrix} x_1^2 - x_2^2 & & \\ & x_2^2 - x_0^2 & \\ & & x_0^2 - x_1^2 \end{pmatrix}$$

in our pencil, passing through x, is n-circumscribed about the conic

$$C := \begin{pmatrix} x_1 - x_2 & & \\ & x_2 - x_0 & \\ & & x_0 - x_1 \end{pmatrix}$$

which is tangent at P_0 to the line T joining x with P_0. Then

$$\begin{aligned} \det(C) &= (x_0 - x_1)(x_1 - x_2)(x_2 - x_0) \\ \det(tD + C) &= \det(C) + t\cdot\det(C)\cdot\{(x_0 + x_1) + (x_1 + x_2) + (x_2 + x_0)\} \\ &\quad + t^2\cdot\det(C)\cdot\{(x_0 + x_1)(x_1 + x_2) + (x_0 + x_1)(x_2 + x_0) \\ &\quad + (x_1 + x_2)(x_2 + x_0)\} \\ &\quad + t^3\cdot\det(C)\cdot(x_0 + x_1)(x_1 + x_2)(x_2 + x_0) \\ \frac{\det(tC + D)}{\det(C)} &= 1 + t\cdot 2s_1 + t^2\cdot(s_1^2 + s_2) + t^3\cdot(s_1 s_2 - s_3)\,. \end{aligned}$$

In particular we find

$$\begin{aligned} A_1 &= s_1 \\ A_2 &= \tfrac{1}{2}s_2 \\ A_3 &= -\tfrac{1}{2}s_3 \\ A_4 &= \tfrac{1}{2}s_1 s_3 - \tfrac{1}{8}s_2^2\,. \end{aligned}$$

This gives the right polynomial f_n for $n \leqq 5$. In the computation of the A_n lots of cancellations take place. The higher order computations are for symbolic manipulation on the computer. So we did not try to evaluate this formula further. However, since all coefficients in the power series expansion are rational, we deduce from it:

(5.2) *The plane curve Π_n is defined over the field of rational numbers. Its equation f_n can be chosen with integer coefficients*

If we fix C in our pencil, then the intersections x of Π_n with the tangent T to C at P_0 determine all the conics $C_{(\lambda:\mu)}$ in the pencil n-circumscribed about C. As Π_n does not pass through the point P_0, the general line T through this point does not touch Π_n in its points of intersection with this curve. Then T intersects Π'_n in precisely $c(n)$ points. This shows

Theorem 1. *Let $\{C_{(\lambda:\mu)}:(\lambda:\mu)\in \mathbb{P}_1\}$ be a pencil of plane conics with four distinct base points. For the general conic C in this pencil there are $c(n)$ different conics $C_{(\mu:\lambda)}$ in the pencil n-circumscribed about C.*

As Π_n does not pass through any of the four base points, the general conic in the pencil does not touch Π_n in its points of intersection with this curve. The lines T connecting such a point with the origin P_0 are all distinct and therefore tangent to as many distinct conics C in the pencil. This proves

Theorem 2. *Let $\{C_{(\lambda:\mu)}:(\lambda:\mu)\in \mathbb{P}_I\}$ be a pencil of conics as in Theorem 1. For the general conic $C_{(\lambda:\mu)}$ in this pencil there are precisely $2\cdot c(n)$ different conics in the pencil n-inscribed into $C_{(\lambda:\mu)}$.*

6 Simultaneously inscribed and circumscribed conics

First we introduce an involution I of the plane. Two points x and $y\in\mathbb{P}_2$ are in involution under I if

1. $x = T\cap C$, where C is a conic in the pencil and T a line through P_0,
2. $y = T^*\cap C^*$, where again C^* is a conic in the pencil and T^* a line through P_0.
3. C^* is tangent to T and T^* is tangent to C at P_0.

(6.1) *The involution I is the Cremona transformation*

$$y_0 = x_1^2 + x_2^2 - x_0^2 + x_0x_1 + x_1x_2 + x_0x_2\ ,$$
$$y_1 = x_2^2 + x_0^2 - x_1^2 + x_0x_1 + x_1x_2 + x_0x_2\ ,$$
$$y_2 = x_0^2 + x_1^2 - x_2^2 + x_0x_1 + x_1x_2 + x_0x_2\ ,$$

based on the three points P_1, P_2, P_3 of the pencil different from P_0.

Proof. The point $P_0 = (1:1:1)$ is a fixed point of the Cremona transformation $(x_0:x_1:x_2)\mapsto(y_0:y_1:y_2)$. Each conic in our pencil therefore is transformed into a line through P_0. All we have to show is that at P_0 the conic and this line are tangent at to each other. But let

$$\alpha x_0 + \beta x_1 + \gamma x_2 = 0, \qquad \alpha+\beta+\gamma = 0\ ,$$

be a line through P_0. Its transform is the conic

$$(\beta+\gamma-\alpha)x_0^2 + (\gamma+\alpha-\beta)x_1^2 + (\alpha+\beta-\gamma)x_2^2 = -2(\alpha x_0^2 + \beta x_1^2 + \gamma x_2^2) = 0\ .$$

□

Next we define the curve $\Pi_n^* := I(\Pi_n)\subset\mathbb{P}_2$. It is a birational image of Π_n. Since Π_n does not pass through the three base points of the Cremona transform I, we have

$$\deg(\Pi_n^*) = 2\cdot\deg(\Pi_n)\ .$$

We are interested in Π_n^* for the following reason: A control point $P = D \cap T_{P_0}(C) \in \Pi_n$ is mapped under I to a point $P^* = T_{P_0}(D) \cap C$. The roles of the conics C and D are interchanged. If we use $Q = P^* \in \Pi_n^*$ as control point, just as we did before with points in Π_n, we get a pair C, D in the pencil such that D is *n-inscribed* into C.

(6.2) *The curve Π_n^* is the locus of points $T_{P_0}(C) \cap D$ where C and D belong to the pencil such that D n-inscribed into C.*

This is nothing sensational, but for the points in the intersection $\Pi_n \cap \Pi_m^*$. These points parametrize pairs C, D of conics in the pencil such that *simultaneously C is n-inscribed into D and D is m-inscribed into C.* Quite amusing situations – they lead to pairs of elliptic curves, whose mutual relation seems interesting to us, although we do not understand it at the moment.

The curve Π_n is singular only in the three coordinate vertices. They lie on the fundamental lines $x_i = -x_j$ of the Cremona transformation I and are transformed into the three base points. Therefore

(6.3) *The curve Π_n^* is singular only in the three points P_1, P_2, P_3. It does not intersect the three lines $x_i = -x_j$ but for these three singularities.*

This means that curves Π_m and Π_n^* never intersect in points which are singular on one of them. So

$$\deg(\Pi_m) \cdot \deg(\Pi_n^*)$$

is the number of their smooth intersection points, counted however with the order of contact as multiplicity. As we are unable to show that these intersections are always transversal, we refrain from giving a detailed statement.

(6.4) *There is a one-to-one correspondence between equivalence classes of*

- *pairs of smooth conics $C, D \subset \mathbb{P}_2$, having four distinct intersection points, such that C at the same time is m-inscribed into and n-circumscribed about D (equivalent under the projective group;)*
- *orbits of points $x \in \Pi_n \cap \Pi_m^*$, not on a line L_i (equivalent under the symmetric group) Σ_3.*

Proof. Given a pair C, D as in the statement, there is a projective transformation mapping their four intersection points onto the four points $(1 : \pm 1 : \pm 1)$ and the two conics onto conics in our pencil. Since C is m-inscribed into D, the tangent T to C at P_0 intersects D in a point $x \in \Pi_m$. Since C is n-circumscribed about D, the tangent T^* to D at P_0 meets C in a point $y \in \Pi_n$. The involution I maps y onto x showing that $x \in \Pi_m \cap \Pi_n^*$.

The projective transformation above is unique up to Σ_4-symmetries. Distinguishing the origin $P_0 = (1:1:1)$ we have Σ_3-symmetries only. This proves the bijection in the statement. □

If we intend n-gons in this statement to be *primitive*, the curve Π_n has to be replaced by Π_n' and Π_m^* by the curve $I(\Pi_m')$. Their intersection number is

$$2 \cdot c(n) \cdot c(m) .$$

If there is no fixed point for Σ_3 among the points of intersection of these two curves, then $\frac{1}{3}\cdot c(n)\cdot c(m)$ is the number of orbits. There are three types of fixed points:

the fixed point $P_0 = (1:1:1)$. It does not lie on any curve Π_n.

the orbit of two points $(1:\omega^k:\omega^{2k})$, $k = 1, 2$, with ω a primitive third root of unity. These two points lie in $\Pi_3 \cap \Pi_3^*$, see the following example.

the points on the lines L_k and L'_k. The curves Π_n and Π_n^* never meet on L'_k, but L_k might contain intersections of Π_n and Π_m^*. For these points the conics degenerate however.

The intersection $\Pi_n \cap \Pi_m^*$ contains at most $2\cdot c(n)\cdot c(m)$ points. The number shrinks perhaps, if we remove points on the three lines L_k, $k = 0, 1, 2$. There remain at most $\frac{1}{3}\cdot c(n)\cdot c(m)$ orbits of points in this intersection, unless $n = m = 3$. In this case one orbit consists of three points, and the computation in the first example below shows that there is another additional orbit. This proves

Theorem 3. *Each smooth conic in the plane $\mathbb{P}_2$ is (up to projective equivalence) simultaneously n-inscribed into and m-circumscribed about*

$$\begin{cases} 2 & \text{if } m = n = 3 \\ \leqq \frac{1}{3}\cdot c(n)\cdot c(m) & \text{if } m \text{ or } n > 3 \end{cases}$$

conics D (meeting C in four distinct points).

We don't know if there are more general properties about the intersections $\Pi_m \cap \Pi_n^*$ which can be proven in this context. So we conclude by computing explicitly the simplest examples. First we need the transforms of the symmetric polynomial s_2 under I. Abbreviating

$$p := s_1^2 - s_2 ,$$

the Cremona transformation I is written

$$y_k = p - 2x_k^2, \quad k = 0, 1, 2 .$$

Therefore

$$I: s_2 \mapsto s_2^* := 3p^2 - 4p(s_1^2 - 2s_2) + 4\sigma_2 = -s_1^4 + 6s_1^2 s_2 - 8s_1 s_3 - s_2^2 .$$

The case $(m, n) = (3, 3)$. The points in $\Pi_3 \cap \Pi_3^*$ are defined by $s_2 = s_2^* = 0$. This is equivalent with

$$s_2 = s_1(8s_3 + s_1^3) = 0 .$$

Now these equations are solved elementarily. We find the following eight points $\left(\omega \text{ a primitive third root of unity, } \eta := \frac{1+\sqrt{5}}{2}, \eta' := \frac{1-\sqrt{5}}{2}\right)$:

- The two points $(1:\omega^k:\omega^{2k})$, $k = 1, 2$.
- The orbit of $(1:\eta:\eta')$ under S_3.

The case $(m, n) = (4, 3)$. The curve Π_4 splits into the three coordinate lines $x_i = 0$. The intersection of Π_3^* with the line $x_0 = 0$ is

$$(x_1 + x_2)^4 - 6(x_1 + x_2)^2 x_1 x_2 + (x_1 x_2)^2 = 0 .$$

Again this is solved elementarily to give

$$(x_0 : x_1 : x_2) = (0 : 1 + \sqrt{2} + \sqrt{-1 + 2\sqrt{2}} : 1 + \sqrt{2} - \sqrt{-1 + 2\sqrt{2}}) .$$

Here x_1 and x_2 can be permuted. For $\sqrt{2}$ we have two signs, but in the same point the same sign must be taken.

The case $(m, n) = (3, 4)$. We just Cremona transform the points from the last case under I. We have

$$s_1 = x_1 + x_2 = 2 + 2\sqrt{2}$$

$$s_2 = x_1 x_2 = 4$$

$$p = s_1^2 - s_2 = 8 + 8\sqrt{2}$$

$$x_1^2 = 2 + 4\sqrt{2} + (2 + 2\sqrt{2})\sqrt{-1 + 2\sqrt{2}}$$

$$x_2^2 = 2 + 4\sqrt{2} - (2 + 2\sqrt{2})\sqrt{-1 + 2\sqrt{2}} .$$

So the transforms are

$$(y_0 : y_1 : y_2) = (2 : \sqrt{2} - 1 + \sqrt{-1 + 2\sqrt{2}} : \sqrt{2} - 1 - \sqrt{-1 + 2\sqrt{2}}) .$$

The case $(m, n) = (4, 4)$. The line $x_0 = 0$ transforms under I into the conic

$$-x_0^2 + x_1^2 + x_2^2 + x_0 x_1 + x_0 x_2 + x_1 x_2 = 0 .$$

The curve Γ_4^* therefore consists of the three transforms of this conic under the symmetric group Σ_3. The intersection of this conic with the line $x_0 = 0$ is the pair $(0 : \omega^k : \omega^{2k})$, $k = 1, 2$. The intersection with the line $x_1 = 0$ is the set of two points $(1 : 0 : \eta)$ and $(1 : 0 : \eta')$.

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