Poncelet theorems

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0. Introduction

The aim of this note is to collect some more or less classical theorems of Poncelet type and to provide them with short modern proofs. Where classical geometers used elliptic functions (or angular functions), we use elliptic curves (or degenerate elliptic curves decomposing into two rational curves). In this way we unify the geometry underlying these Poncelet type statements.

Our starting point is a space Poncelet theorem for two quadrics in $\mathbb{P}_3$ (Sect. 1). This seems to have been observed first by Weyr [12] p. 28, and amplified by Griffiths and Harris [5].

The classical Poncelet theorem (cf. [6]), a statement on two conics in the plane $\mathbb{P}_2$, follows from Weyr's space Poncelet theorem, if one of the quadrics is taken to be a cone, see Sect. 1.

Gerbaldi [4] gave formulas counting the number of conics in a pencil which are in Poncelet position with respect to a fixed conic in the pencil. We show that his formulas are simple consequences of the space Poncelet theorem (Sect. 3).

In Sect. 5 we evaluate explicitly the space Poncelet condition for two quadrics of revolution about the same axis.

We show that theorems such as Emch's theorem on circular series [3] and (a complex-projective version of) the 'zig-zag' theorem [2] can be understood by considering torsion points on elliptic curves (see Sects. 7 and 8). Further, we prove a Poncelet version of the Money-Coutts theorem (Sect. 9).

Although for most of the contents of this paper only the presentation is new, it seems worth–while to us to consider Poncelet type theorems from a modern geometric point of view. In this spirit the equations of modular curves given in [1] have been transformed by N. Hitchin [7] into solutions of the Painlevé–VI–equation. And E. Previato relates Poncelet theorems to integrable Hamiltonian systems and billiards [10].

Conventions. The base field always is the field $\mathbb{C}$ of complex numbers.

If we mention circles, quadrics of revolution, or spheres, we mean the corresponding varieties over $\mathbb{C}$.

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1. Weyr's Poncelet theorem in $\mathbb{P}_3$

Let $Q_1, Q_2 \subset \mathbb{P}_3$ be quadrics of ranks ≥ 3 such that their intersection curve $E = Q_1 \cap Q_2$ is either a smooth elliptic curve or the union of two conics C_1, C_2 meeting in two distinct points. We fix rulings R_1 on Q_1 and R_2 on Q_2.

(1.1) Theorem ([12] p. 28, [8] p. 13). *Suppose that there exists a closed sequence of distinct lines $L_1, \ldots, L_{2n}, L_{2n+1} = L_1$ such that the line L_i belongs to R_1 resp. R_2, if i is odd resp. even, and such that consecutive lines L_i, L_{i+1} intersect each other. Then there are such closed sequences of length $2n$ through any point on $Q_1 \cap Q_2$.*

Proof. The rulings R_1, R_2 define involutions ι_1, ι_2 on E interchanging the two intersection points of E with a line in R_1 resp. R_2. Let $t : E \longrightarrow E$ be the composition $\iota_2\iota_1$ and let $e := L_{2n} \cap L_1$. Then L_{2k-1} is the unique line in R_1 through $t^{k-1}e$ and L_{2k} is the unique line in R_2 through $\iota_1 t^{k-1}e$ for $k \geq 1$. The closedness $L_{2n+1} = L_1$ implies $t^n(e) = e$.

First we consider the case that E is smooth elliptic. The involutions ι_1, ι_2 have fixed points, so their composition t is a translation on E. But then $t^n(e) = e$ and the assumption that the lines $L_1, \ldots, L_n$ are distinct imply that t is of order n, hence the assertion.

Now assume $E = C_1 \cup C_2$ is the union of two conics. The involutions ι_1 and ι_2 interchange these conics. So $t(C_i) = C_i$, $i = 1, 2$ and the two intersections of C_1 and C_2 are fixed points of t. Therefore t induces two automorphisms $t_i : \mathbb{C}^* \cong C_i \setminus (C_1 \cap C_2) \longrightarrow C_i \setminus (C_1 \cap C_2) \cong \mathbb{C}^*$. If e lies on C_1, say, then $t^n(e) = e$ implies that t_1 is the multiplication by a primitive n-th root of unity, i.e. its order is n. But then t_2 is of order n as well, because of $t_2 = \iota_2 t_1^{-1} \iota_2$. □

Theorem 1.1 also holds for two different rulings R_1 and R_2 on the same smooth quadric $\simeq \mathbb{P}_1 \times \mathbb{P}_1$, if E is a curve on this quadric of bidegree $(2, 2)$, either smooth elliptic or the union of two rational curves meeting in two distinct points.

The proof is literally the same.

We say that the pair of rulings (R_1, R_2) *satisfies the Poncelet-n-condition* if the automorphism $t : E \longrightarrow E$ defined above is of order n. A priori this is a property of the *ordered* pair (R_1, R_2). But the automorphism associated to (R_2, R_1) is just t^{-1}, thus:

(1.2) *The pair of rulings (R_1, R_2) satisfies the Poncelet-n-condition if and only if (R_2, R_1) does.*

Already Hurwitz [8], p. 13, observed that Theorem 1.1 implies the usual Poncelet theorem in $\mathbb{P}_2$.

In fact, if one of the quadrics, Q_1, is smooth and the other one, Q_2, is a cone with its top P_0 not on $E = Q_1 \cap Q_2$ and E is smooth elliptic, then the three-dimensional Poncelet theorem is equivalent to Poncelet's theorem for two conics in the plane. To see this, we denote by $\pi_i : Q_i \longrightarrow \mathbb{P}_2$ the projections from P_0. The morphism π_1 is of degree 2, ramified over a smooth conic $C \subset \mathbb{P}_2$. The image of π_2 is another smooth conic $D \subset \mathbb{P}_2$ in general position with respect to C. We have:

(1.3) Proposition. *The quadrics Q_1 and Q_2 are in Poncelet-n-position if and only if the conic C is n-inscribed into D.*

We say: C is n–inscribed into D, if there is a polygon consisting of n tangents to C with its vertices on D, [1].

Proof. We choose an isomorphism $Q_1 \cong \mathbb{P}_1 \times \mathbb{P}_1$ such that the ruling R_1 is parametrized by the first factor. Let $\Delta = \pi^{-1}C \subset \mathbb{P}_1 \times \mathbb{P}_1$ be the ramification curve. It is a curve of bidegree $(1,1)$ on $\mathbb{P}_1 \times \mathbb{P}_1$ and induces an isomorphism between both copies of $\mathbb{P}_1$ such that Δ is the diagonal. We consider the map

$$\begin{array}{ccc} Q_1 & \longrightarrow & \mathbb{P}_2 \times C^* \\ (a,b) & \longmapsto & (u,T) \end{array}$$

where $u := \pi_1(a,b)$ and $T := \pi_1(a \times \mathbb{P}_1)$. So T is the tangent from u to C meeting C in the point $\pi_1(a,a)$. The projection π_1 restricts to a map $E \longrightarrow D$ of degree 2. Over a point $u \in D$ we have points $(a,b), (b,a) \in E$. Now we determine the map on the pairs (u,T) induced by t. We start with a smooth point $P = (a,b)$ on E. The line $L_1 = a \times \mathbb{P}_1 \in R_1$ through P meets E in another point $P' = (a,b')$. This means that we pass from the pair (u,T) to (u',T), where $u' := \pi_1(a,b')$ is the second intersection point of T and D. The image point $P'' := t(P)$ is the second intersection point of the line $L_2 \in R_2$ through P' with E. Thus we have $P'' = (b',a)$, because points in the same ray of Q_2 lie in the same fibre of the projection π_2. Passing from the point (a,b') to (b',a) means passing from the pair (u',T) to (u',T'), where T' is the second tangent to C through the point u'. So we see that the map

$$t : (u,T) \longmapsto (u',T')$$

just describes the Poncelet process for the pair of conics C, D. This implies the assertion. □

We should point out here, how Theorem 1.1 relates to the Poncelet theorem in space considered by Griffiths and Harris in [5] on finite polyhedra both inscribed and circumscribing two smooth quadrics Q_1 and Q_2: Such a polyhedron exists if and only if both of the pairs R_1, R_2 and R'_1, R'_2 satisfy a Poncelet condition in the sense above, where R_i, R'_i are the rulings on the dual quadrics Q_i^*, $i = 1, 2$.

2. Poncelet pairs in a pencil

Let Q_1, Q_2 be quadrics of rank ≥ 3 such that the pencil $\lambda_1 Q_1 + \lambda_2 Q_2$ generated by them is generic. This means that its base locus $E = Q_1 \cap Q_2$ is smooth, or equivalently that the discriminant $d(\lambda_1, \lambda_2) = det(\lambda_1 Q_1 + \lambda_2 Q_2)$ of the pencil has no multiple roots. The four roots of $d(\lambda_1, \lambda_2)$ correspond to the cones in the pencil, i.e. to those quadrics carrying only one ruling. Let $M \longrightarrow \mathbb{P}_1$ be the double cover of $\mathbb{P}_1$ branched over the roots of $d(\lambda_1, \lambda_2)$. The points of the elliptic curve M can be identified with the rulings on the quadrics of the given pencil. Each ruling R in M defines an involution $I_R : E \longrightarrow E$ with fixed points. Choosing an origin in E, we can write I_R as $x \longmapsto -x + a$ with a unique point $a \in E$, so we obtain a map

$$\begin{aligned} \Phi : M &\longrightarrow E \\ R &\longmapsto a\ . \end{aligned}$$

(2.1) Proposition. *Φ is an isomorphism of groups (if the origin of M is chosen appropriately).*

Proof. It suffices to show that Φ is injective. So let $R_1, R_2 \in M$ such that $\Phi(R_1) = \Phi(R_2)$, i.e. such that the involutions I_{R_1}, I_{R_2} coincide. Let $P \in E$ be a point which is not fixed under I_{R_1} or I_{R_2}. Because of $I_{R_1} = I_{R_2}$ the lines

$$\overline{P, I_{R_1}(P)} \in R_1$$
$$\overline{P, I_{R_2}(P)} \in R_2$$

are equal. Varying the point P in E we conclude $R_1 = R_2$. □

This gives the following corollary which will be applied in Sect. 3.

(2.2) Corollary. *a) Two rulings R_1, R_2 in the pencil satisfy the Poncelet-n-condition, if and only if the point $R_2 - R_1 \in M$ is a primitive n-torsion point.*

b) For a fixed ruling $R_1 \in M$ there are $T(n)$ rulings $R_2 \in M$ such that R_1, R_2 satisfy the Poncelet-n-condition. Here $T(n)$ denotes the number of primitive n-torsion points on an elliptic curve.

This corollary is independent of the choice of origin on M, if one interprets $R_2 - R_1$ as the translation on M mapping R_1 to R_2.

In general the Poncelet property depends on the choice of the rulings R_1, R_2 on Q_1, Q_2. However if it holds for R_1 and R_2, then it also holds for the complementary rulings R_1' on Q_1 and R_2' on Q_2. In fact, $R_2 - R_1 = -(R_2' - R_1')$, independently of the choice of origin on M. This is even true if one of the quadrics, say Q_2 is a cone and $R_2' = R_2$:

(2.3) *Let Q_1 be smooth and let Q_2 be a cone. We denote by R_1, R_1' the rulings on Q_1 and by R_2 the unique ruling on Q_2. Then R_1', R_2 satisfy the Poncelet-n-condition if and only if R_1, R_2 do.*

3. Gerbaldi's formula for the number of inscribed conics

Gerbaldi [4] considered the invariant of two conics C and $D \subset \mathbb{P}_2$ vanishing if C is n–inscribed into D. Using elliptic functions and continued fractions he showed:

(3.1) Theorem ([4], p. 103). *This invariant is of degree $\frac{1}{2}T(n)$ in C and of degree $\frac{1}{4}T(n)$ in D.*

Here we do not want to make his assertion precise. Instead we prove the following two assertions, together equivalent with Theorem 3.1. Ignorantly of Gerbaldi's paper they were shown in [1] using a rational elliptic surface.

(3.2) Theorem. *Let $\lambda C + \mu D$, $(\lambda : \mu) \in \mathbb{P}_1$, be a generic pencil of conics in $\mathbb{P}_2$. Then the number of conics in the pencil, which are n-inscribed into D is $\frac{1}{2}T(n)$.*

Proof. 1) There is a smooth quadric $Q_1 \subset \mathbb{P}_3$ and a cone $Q_2 \subset \mathbb{P}_3$ such that

i) the branch locus of the projection $Q_1 \longrightarrow \mathbb{P}_2$ from the top P_0 of Q_2 is C,

ii) the image of Q_2 under P_0-projection is D, and

iii) the pencil generated by Q_1 and Q_2 is generic.

In fact, in suitable homogeneous coordinates x, y, t on $\mathbb{P}_2$ and x, y, z, t on $\mathbb{P}_3$ the conics C and D are given by equations

$$C : x^2 + y^2 + t^2 = 0, \qquad D : \alpha x^2 + \beta y^2 + \gamma t^2 = 0$$

with $\alpha, \beta, \gamma \in \mathbb{C}$. We choose Q_2 to be the cone

$$Q_2 : \alpha x^2 + \beta y^2 + \gamma t^2 = 0$$

with top $P_0 = (0 : 0 : 1 : 0)$ and Q_1 to be the smooth quadric

$$Q_1 : x^2 + y^2 + z^2 + t^2 = 0 \ .$$

Now conditions i) and ii) are obviously satisfied. For condition iii) note that the discriminant of the pencil $\lambda_1 Q_1 + \lambda_2 Q_2$,

$$d(\lambda_1, \lambda_2) = \lambda_1(\lambda_1 + \lambda_2\alpha)(\lambda_1 + \lambda_2\beta)(\lambda_1 + \lambda_2\gamma) \ ,$$

has no multiple roots, since the pencil $\lambda C + \mu D$ is generic by assumption.

2) Let $Q_1, Q_2 \subset \mathbb{P}_3$ be quadrics with the properties i),ii),iii) above. We claim that the branch locus of a smooth quadric $Q_{\lambda_1,\lambda_2} = \lambda_1 Q_1 + \lambda_2 Q_2$ under P_0-projection is a conic in the pencil $\lambda C + \mu D$. It is enough to show that the branch divisors of the P_0-projections $Q_{\lambda_1,\lambda_2} \longrightarrow \mathbb{P}_2$ vary in a pencil. Now, a point $P \in Q_{\lambda_1,\lambda_2}$ is a branch point, if the line $\overline{PP_0}$ touches Q_{λ_1,λ_2} in P, i.e. if $P_0^t Q_{\lambda_1,\lambda_2} P = 0$ (in matrix notation). So the branch divisor is just the intersection of Q_{λ_1,λ_2} with the polar $P_0^t Q_{\lambda_1,\lambda_2}$ of the point P_0. But

$$P_0^t Q_{\lambda_1,\lambda_2} = P_0^t(\lambda_1 Q_1 + \lambda_2 Q_2) = \lambda_1 P_0^t Q_1$$

shows that this polar is the same for all quadrics in the pencil. So the branch divisors vary in a pencil on this polar.

3) If a quadric Q_{λ_1,λ_2} of the pencil is smooth, then according to Proposition 1.3 the quadrics Q_{λ_1,λ_2} and Q_2 are in Poncelet-n-position, if and only if the branch locus of Q_{λ_1,λ_2} is n-inscribed into D. If Q_{λ_1,λ_2} is a cone, then Q_{λ_1,λ_2} and Q_2 are certainly not in Poncelet-n-position, because their rulings are halfperiods on the elliptic curve M parametrizing the rulings in the pencil. We conclude that the number of conics in the pencil $\lambda C + \mu D$, which are n-inscribed into D equals the number of quadrics Q_{λ_1,λ_2} such that Q_{λ_1,λ_2} and Q_2 are in Poncelet-n-position. But this number is just half the number of rulings $R \in M$ such that R, R_2 satisfy the Poncelet-n-condition. Now the assertion of the theorem follows by Corollary 2.2. □

(3.3) Theorem. *Let $\lambda C + \mu D$ be a general pencil of conics in $\mathbb{P}_2$. Then the number of conics in this pencil, n–circumscribed about D is $\frac{1}{4}T(n)$.*

Proof. A conic $\lambda C + \mu D$ is n–circumscribed about D if and only if the dual conic $(\lambda C + \mu D)^*$ is n–inscribed into D^*. The dual pencil $(\lambda C + \mu D)^*$ is parametrized by a smooth conic Γ in the space $\mathbb{P}_5$ of all conics.

Denote by $H \subset \mathbb{P}_5$ the hypersurface parametrizing conics n–circumscribed about D. By Theorem 3.2

$$\deg(H) = \frac{1}{2}\Gamma.H = \frac{1}{4}T(n).$$

This shows that the general pencil $\lambda C + \mu D$ meets H in $\frac{1}{4}T(n)$ points. □

4. A Poncelet theorem on three conics

As an application of the three-dimensional Poncelet theorem we now prove a Poncelet theorem on three conics.

Let $C, C_1, C_2 \subset \mathbb{P}_2$ be three smooth conics in a generic pencil. Let P_1 be an arbitrary point on C. There are two tangent lines to C_1 through P_1. We choose one of them, T_1 say, and define P_2 to be its second point of intersection with C. Next, we choose a tangent T_2 to C_2 through P_2. We will now describe, how the data P_1, T_1, T_2 determine a sequence $(T_i)_{i\geq 1}$ of lines such that for $i \geq 1$ the line T_i is tangent to C_1

resp. C_2, if i is odd resp. even, and such that the intersection points of consecutive lines T_i, T_{i+1} lie on C.

To this end, we choose (as in the proof of Theorem 3.2) smooth quadrics Q_1, Q_2 and a cone Q in a generic pencil in $\mathbb{P}_3$ such that, denoting the projection $\mathbb{P}_3 \dashrightarrow \mathbb{P}_2$ from the top of Q by π,

i) the branch locus of $\pi|_{Q_i} : Q_i \longrightarrow \mathbb{P}_2$ is C_i, $i = 1, 2$, and
ii) the image of Q under π is C.

Then we have $P_1 = \pi(e)$ for some point e on the elliptic curve $E = Q_1 \cap Q_2$. The tangent T_1 is the image $\pi(L_1)$ of a line L_1 on Q_1 through e. Let R_1 be the ruling on Q_1 containing L_1 and let ι_1 be the associated involution on E. So $P_2 = \pi(\iota_1 e)$ and T_2 is the image $\pi(L_2)$ of a line L_2 on Q_2 through $\iota_1 e$. Let R_2 be the ruling on Q_2 containing L_2 and ι_2 its associated involution. Now the tangents T_i, $i \geq 3$, are defined to be the images of the lines L_i, where for $k \geq 2$

$$\begin{aligned} L_{2k-1} &:= \text{ the line in } R_1 \text{ through the point } (\iota_2\iota_1)^{k-1}e \\ L_{2k} &:= \text{ the line in } R_2 \text{ through the point } \iota_1(\iota_2\iota_1)^{k-1}e \ . \end{aligned}$$

The intuition here is that the choice of the tangents T_i, $i \geq 3$, is compatible with the choices made for T_1 and T_2.

Now suppose that the tangent sequence (T_i) *closes after n steps*, i.e. $T_{2n+1} = T_1$. This occurs if and only if the translation $t := \iota_2\iota_1$ is of order n. Since this is independent of the starting point P_1, we have:

(4.1) Theorem. *Suppose C, C_1, C_2 are smooth conics in a generic pencil in $\mathbb{P}_2$ admitting a tangent sequence (in the sense above) which closes after n steps. Then they admit such sequences starting with any point on C.*

The considerations above show slightly more: The existence of a closed tangent sequence is actually a property of the pair C_1, C_2 only, so it is not only independent of the starting point on C, but even on the choice of C within the pencil generated by C_1 and C_2.

5. The Poncelet condition for quadrics of revolution

The aim of this section is to give an explicit formula for the Poncelet-n-condition on two quadrics of revolution.

To begin with, let q, q' be quadrics in $\mathbb{P}_1$, given by equations

$$q(z,t) = az^2 + bzt + ct^2, \qquad q'(z,t) = a'z^2 + b'zt + c't^2 \ .$$

The pair (q, q') has three quasi-invariants under the $\mathrm{PGL}(2,\mathbb{C})$-action: the *discriminants* $D := b^2 - 4ac$, $D' := b'^2 - 4a'c'$ and the *jacobian* $J := 2(ac' + a'c) - bb'$. We have:

(5.1) *If $D, D' \neq 0$, then up to the* $\mathrm{PGL}(2,\mathbb{C})$*-action the pair (q, q') is uniquely determined by its invariants D, D' and J.*

Proof. Because of $D' \neq 0$ we may assume $q'(z,t) = zt$, i.e. $a' = 0$, $c' = 0$ and $b' = 1$. Then we have $D' = 1$ and $J = -b$. So b is determined by J and the product ac is determined by $D = b^2 - 4ac$. For any $\alpha \in \mathbb{C}$ the projective transformation $(z : t) \longmapsto (\alpha z : \frac{t}{\alpha})$ leaves the invariants D, D', J unchanged. It transforms the quadric q into

$$\alpha^2 a z^2 + bzt + \frac{c}{\alpha^2} t^2 ,$$

so by a suitable choice of α we can change a to any nonzero value. □

In terms of the invariants D, D', J the discriminant $d(\lambda, \mu)$ of the pencil $\lambda q + \mu q'$, $(\lambda : \mu) \in \mathbb{P}_1$ reads

$$d(\lambda, \mu) = -\frac{\lambda^2}{4} D + \frac{\lambda\mu}{2} J - \frac{\mu^2}{4} D' .$$

Next we consider the quadrics

$$Q_{a,b,c} : x^2 + y^2 = az^2 + bzt + ct^2, \qquad a, b, c \in \mathbb{C}$$

in $\mathbb{P}_3$ with homogeneous coordinates $(x : y : z : t)$. Such a quadric $Q_{a,b,c}$ is singular (a cone), if $b^2 = 4ac$. The intersection curve of two quadrics Q_{a_1,b_1,c_1} and Q_{a_2,b_2,c_2} consists of two circles

$$C_j : x^2 + y^2 = a_1 z_j^2 + b_1 z_j t_j + c_1 t_j^2$$

in the planes $zt_j = tz_j$, $j = 1, 2$, where $(z_1 : t_1)$ and $(z_2 : t_2)$ are the roots of the equation

$$(a_1 - a_2)z^2 + (b_1 - b_2)zt + (c_1 - c_2)t^2 = 0 .$$

In case $(b_1 - b_2)^2 = 4(a_1 - a_2)(c_1 - c_2)$ the two circles coincide, i.e. the quadrics touch along one circle. Otherwise the circles C_1, C_2 meet in the two points $(1 : i : 0 : 0), (i : 1 : 0 : 0)$.

Now we give an explicit formula for the Poncelet-n-condition on two quadrics Q_{a_1,b_1,c_1} and Q_{a_2,b_2,c_2}. This is best expressed in terms of the invariants of the binary quadrics $q_{a_i,b_i,c_i} := a_i z^2 + b_i zt + c_i t^2$.

(5.2) Proposition. *Let Q_{a_1,b_1,c_1} and Q_{a_2,b_2,c_2} be two quadrics as above with reduced intersection curve (two circles) and let D_1, D_2, J_{12} be the discriminants and the jacobian of the associated binary quadrics. Then the following statements are equivalent:*

i) Q_{a_1,b_1,c_1} and Q_{a_2,b_2,c_2} are in Poncelet-n-position.

ii) There is an integer k, $(k, n) = 1$, such that

$$\left(J_{12}(1 + \cos\frac{2\pi k}{n}) + D_1 + D_2 \right)^2 = D_1 D_2 \left(1 - \cos\frac{2\pi k}{n} \right)^2$$

Proof. Let C_1, C_2 be the circles of intersection of the two quadrics and let $zt_j = tz_j$, $j = 1, 2$, be the associated planes. It will be enough to consider the case that $t_1 = t_2 = 1$. We may then calculate in affine coordinates x, y, z.

To begin with, let

$$P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2), \quad P_1' = (x_1', y_1', z_1)$$

be three points such that the line $\overline{P_1 P_2}$ lies on Q_1 and $\overline{P_2 P_1'}$ lies on Q_2. Now, $\overline{P_1 P_2}$ lies on Q_1 if and only if the points P_1, P_2 belong to Q_1, i.e.

$$x_j^2 + y_j^2 = a_1 z_j^2 + b_1 z_j + c_1$$

and if the point $(P_1 + P_2)/2$ lies on Q_1, i.e.

$$\left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{y_1 + y_2}{2}\right)^2 = a_1 \left(\frac{z_1 + z_2}{2}\right)^2 + b_1 \left(\frac{z_1 + z_2}{2}\right)^2 + c_1 \ .$$

Because of $P_1, P_2 \in Q_1$ this is equivalent to

$$x_1 x_2 + y_1 y_2 = a_1 z_1 z_2 + \frac{b_1}{2}(z_1 + z_2) + c_1 =: g_1 \ . \tag{1}$$

Similarly the line $\overline{P_2 P_1'}$ lies on Q_2 if and only if

$$x_1' x_2 + y_1' y_2 = a_2 z_1 z_2 + \frac{b_2}{2}(z_1 + z_2) + c_2 =: g_2 \ . \tag{2}$$

Now we come to the Poncelet condition. The automorphisms of order n on $C_1 \setminus \{(1 : i : 0 : 0), (i : 1 : 0 : 0)\} \cong \mathbb{C}^*$ are induced by the maps

$$\rho_k : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos \frac{2\pi k}{n} & -\sin \frac{2\pi k}{n} \\ \sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3}$$

where $(k, n) = 1$. So the two quadrics are in Poncelet-n-position if and only if $\rho_k(x_1, y_1) = (x_1', y_1')$ for some k with $(k, n) = 1$. Without loss of generality we may assume $y_1 = 0$. Then conditions (1), (2) and (3) read

$$\begin{aligned} x_1' &= x_1 \cos \tfrac{2\pi k}{n} && (4) \\ -y_1' &= x_1 \sin \tfrac{2\pi k}{n} && (5) \\ x_1 x_2 &= g_1 && (6) \\ x_1' x_2 + y_1' y_2 &= g_2 && (7) \end{aligned}$$

Inserting the first three equations into the last one we obtain

$$g_1 \cos \tfrac{2\pi k}{n} - x_1 y_2 \sin \tfrac{2\pi k}{n} = g_2 \ . \tag{8}$$

and from the equations (6) and (8) we get

$$x_2^2 + y_2^2 = \left(\frac{g_1}{x_1}\right)^2 + \left(\frac{g_1 \cos \frac{2\pi k}{n} - g_2}{x_1 \sin \frac{2\pi k}{n}}\right)^2 \ .$$

Now, upon using the equation of Q_1 we can replace the left hand side by $a_1 z_2^2 + b_1 z_2 + c_1$ and we can substitute x_1^2 on the right hand side by $a_1 z_1^2 + b_1 z_1 + c_1$. In this way we arrive at the equation

$$g_1^2 - 2g_1 g_2 \cos \tfrac{2\pi k}{n} + g_2^2 = p \sin^2 \tfrac{2\pi k}{n} \tag{9}$$

where $p := (a_1 z_1^2 + b_1 z_1 + c_1)(a_1 z_2^2 + b_1 z_2 + c_1)$. Now we can use that z_1 and z_2 are the roots of the equation $(a_1 - a_2)z^2 + (b_1 - b_2)z + (c_1 - c_2) = 0$ to write g_1, g_2 and p in terms of the coefficients $a_1, b_1, c_1, a_2, b_2, c_2$. This finally allows to express (9) in the invariants D_1, D_2, J_{12}. We omit the details. □

6. Circles in the projective plane

In the previous sections we considered Poncelet properties of conics and quadrics related to the three-dimensional Poncelet theorem. Now we turn to the study of closing theorems in circle geometry.

A *circle* is a conic $C \subset \mathbb{P}_2$ passing through the two circular points $(1 : \pm i : 0)$. Its equation is of the form

$$a(x^2 + y^2) + 2bxz + 2cyz + dz^2 = 0 \ ,$$

where $(a : b : c : d) \in \mathbb{P}_3$. The *discriminant* of the circle C is

$$\det \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ b & c & d \end{pmatrix} = a(ad - b^2 - c^2) \ .$$

The quadratic form

$$q(C) := ad - b^2 - c^2$$

is the basic invariant. A circle C with $q(C) = 0$ consists of two lines each passing through a circular point. These are the *null-circles*. If we have $a = 0$, then C decomposes into two lines, one of which passes through the two circular points. In this case we say that C is a *line*.

The quadratic invariant q associates with each circle C two basic surfaces in the space of circles $\mathbb{P}_3$:

The polar plane. The polar plane of a circle C with respect to the quadric q is

$$q(C, C') = \tfrac{1}{2}(ad' + da') - (bb' + cc') \ .$$

(6.1) Proposition ([9], p. 32). *We have $q(C, C') = 0$ if and only if the circles C, C' intersect orthogonally, i.e. if they have a point of intersection P such that the tangent vectors of C and C' in P are orthogonal with respect to the canonical bilinear form on $\mathbb{C}^2$.*

Proof. We may use affine coordinates x, y. The tangent vector of C at the point (x, y) is

$$\begin{pmatrix} \frac{\partial C}{\partial x} \\ \frac{\partial C}{\partial y} \end{pmatrix} = \begin{pmatrix} 2ax + 2b \\ 2ay + 2c \end{pmatrix} ,$$

so C and C' intersect orthogonally if and only if there is a point (x, y) such that the following equations are satisfied.

$$\begin{aligned} a(x^2 + y^2) + 2bx + 2cy + d &= 0 \\ a'(x^2 + y^2) + 2b'x + 2c'y + d' &= 0 \\ (2ax + 2b)(2a'x + 2b') + (2ay + 2c)(2a'y + 2c') &= 0 \end{aligned}$$

Elimination of x, y from these equations yields $q(C, C') = 0$, as claimed. □

The tangent cone. This is the variety of circles touching a given circle. We have:

(6.2) Proposition ([9], p. 32). *a) The circles C' touching a given circle C are parametrized by the quadric surface*

$$Q_C : q(C)q(C') - q(C, C')^2 = 0$$

in the space of circles.

b) If C is smooth, then Q_C is a cone with top $C \in \mathbb{P}_3$. If C is a null-circle, then Q_C is a double plane.

Proof. a) The circles C, C' touch if there is a point (x, y) such that

$$C(x, y) = C'(x, y) = 0 dC(x, y) = \lambda \cdot dC'(x, y)$$

for some $\lambda \in \mathbb{C}^*$. By eliminating x, y and λ from these equations we find

$$\begin{aligned} &-\tfrac{1}{4}a^2d'^2 + \tfrac{1}{2}ada'd' - adb'^2 - adc'^2 + abb'd' + acc'd' - \tfrac{1}{4}d^2a'^2 + bda'b' + cda'c' \\ &-b^2a'd' + b^2c'^2 - 2bcb'c' - c^2a'd' + c^2b'^2 = 0 , \end{aligned}$$

which is equivalent to $q(C)q(C') - q(C, C')^2 = 0$.

b) An obvious calculation shows that $C = (a : b : c : d)$ is a singular point of Q_C. Further, we find $\operatorname{rank} Q_C = 1$ if $q(C) = 0$. □

Now we consider the variety of all circles C, which touch two given (smooth) circles C_1, C_2. This is the intersection curve of the two cones

$$\begin{aligned} Q_{C_1} : q(C_1)q(C) - q(C_1, C)^2 &= 0 \\ Q_{C_2} : q(C_2)q(C) - q(C_2, C)^2 &= 0 . \end{aligned}$$

(6.3) Proposition. *a) The intersection of the cones Q_{C_1}, Q_{C_2} consists of two conics lying in the planes*

$$\Pi^{\pm}_{C_1,C_2} := \sqrt{q(C_2)}q(C_1,C) \pm \sqrt{q(C_1)}q(C_2,C) \ .$$

b) The intersection points of the two conics are the null-circles associated to the intersection points of C_1 and C_2.

In the sequel the two conics above will be referred to as the two *families of circles touching C_1 and C_2.*

Proof. a) The cones Q_{C_1}, Q_{C_2} span the pencil of quadrics

$$(\lambda q(C_1) + \mu q(C_2))q(C) - \lambda q(C_1,C)^2 - \mu q(C_2,C)^2 \qquad , (\lambda : \mu) \in \mathbb{P}_1 \ .$$

This pencil contains the quadric

$$q(C_2)Q_{C_1} - q(C_1)Q_{C_2} = q(C_2)q(C_1,C)^2 - q(C_1)q(C_2,C)^2 \ ,$$

which splits in the two planes $\Pi^{\pm}_{C_1,C_2}$.

b) Let C be a null-circle associated to C_1, C_2. It touches both C_1 and C_2, so

$$q(C_1)q(C) - q(C_1,C)^2 = q(C_2)q(C) - q(C_2,C)^2 = 0 \ .$$

Because of $q(C) = 0$ this implies $q(C_1,C) = q(C_2,C) = 0$, so C lies on both planes $\Pi^{\pm}_{C_1,C_2}$. □

We will also need the following degenerate case.

(6.4) Lemma. *If two circles C_1, C_2 touch each other, then the tangent cones Q_{C_1}, Q_{C_2} touch along a line.*

Proof. We may assume that the circles have equations

$$C_1 : x^2 + y^2 + z^2 = 0 C_2 : a(x^2 + y^2) + 2bxz + 2cyz + dz^2 = 0 \ .$$

Let P be the pencil of circles touching C_1 in its point of contact with C_2. Thus P is just the line joining the vertices $p_1 = (1 : 0 : 0 : 1)$ and $p_2 = (a : b : c : d)$ of the two cones. We calculate the tangent planes of Q_{C_1} and Q_{C_2} in the points $\lambda_1 p_1 + \lambda_2 p_2$ of P:

$$\begin{aligned} Q_{C_1}(\lambda_1 p_1 + \lambda_2 p_2) &= \lambda_2 Q_{C_1} p_2 = \tfrac{\lambda_2}{4}(d - a : -4b : -4c : a - d) \\ Q_{C_2}(\lambda_1 p_1 + \lambda_2 p_2) &= \lambda_1 Q_{C_2} p_1 = \\ &\tfrac{\lambda_1}{4}(-d^2 - 2b^2 - 2c^2 + ad : 2b(a+d) : 2c(a+d) : -a^2 - 2b^2 - 2c^2 + ad) \end{aligned}$$

Now C_1, C_2 touch each other, i.e. $C_2 \in Q_{C_1}$, i.e. $a^2 + d^2 + 4b^2 + 4c^2 - 2ad = 0$. This shows that the tangent planes coincide along P. □

In Sect. 9 we will have to work with the tangent cones of three given circles C_1, C_2, C_3. The following property of their mutual intersection will turn out to be crucial.

(6.5) Lemma. *If the signs* $\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23} \in \{\pm 1\}$ *are chosen such that* $\varepsilon_{12}\varepsilon_{13}\varepsilon_{23} = -1$*, then the planes* $\Pi^{\varepsilon_{12}}_{C_1,C_2}, \Pi^{\varepsilon_{13}}_{C_1,C_3}, \Pi^{\varepsilon_{23}}_{C_2,C_3}$ *have a line in common.*

Notice: Changing the sign of a square root $\sqrt{q(C_i)}$ does not influence the condition $\varepsilon_{12}\varepsilon_{13}\varepsilon_{23} = -1$.

Proof. The pencil spanned by $\Pi^{\varepsilon_{12}}_{C_1,C_2}$ and $\Pi^{\varepsilon_{13}}_{C_1,C_3}$ contains the plane

$$\frac{\sqrt{q(C_2)}}{\sqrt{q(C_1)}}\Pi^{\varepsilon_{13}}_{C_1,C_3} - \frac{\sqrt{q(C_3)}}{\sqrt{q(C_1)}}\Pi^{\varepsilon_{12}}_{C_1,C_2} = \varepsilon_{13}\sqrt{q(C_2)}q(C_3,C) - \varepsilon_{12}\sqrt{q(C_3)}q(C_2,C) \ ,$$

which equals $\Pi^{\varepsilon_{23}}_{C_2,C_3}$, if $\varepsilon_{12}\varepsilon_{13}\varepsilon_{23} = -1$. □

7. Emch's and Steiner's theorems on circular series

In this section we consider two classical theorems on circular series. Our aim here is to show that one obtains short proofs by considering an elliptic resp. rational curve underlying the closing mechanism.

Let $C, C_1, C_2 \subset \mathbb{P}_2$ be smooth circles in general position and let F be one of the two families of circles touching C_1 and C_2.

We start with a smooth circle $S_1 \in F$ and choose one of its points of intersection with C, P_1 say. The pair $(S_1, P_1) \in F \times C$ determines a second pair as follows. There are two circles in F through P_1, namely the points of intersection of F with the plane Π_{P_1} in the space of circles consisting of all circles through P_1. Let $F \cap \Pi_{P_1} = \{S_1, S_2\}$. The second circle S_2 in turn intersects C in P_1 and in a second point P_2. We set $t(S_1, P_1) := (S_2, P_2)$.

(7.1) Theorem (Emch [3]). *Suppose* $t^n(S_1, P_1) = (S_1, P_1)$ *holds for some pair* (S_1, P_1) *such that* $P_1 \in C$*. Then this holds for any such pair.*

Proof. Consider the incidence curve

$$E := \{(S, P) \in F \times C \mid P \in S\}$$

Denoting by π_1, π_2 the projections onto F resp. C, we have for $S \in F$

$$\pi_2\pi_1^{-1}(S) = S \cap C = Q + Q' + P + P' \ ,$$

where Q, Q' are the infinitely far points on C and $P, P' \in C$. For $P \in C$ we have $\pi_1\pi_2^{-1}(P) = F \cap \Pi_P$, where Π_P is the plane in the space of circles consisting of the circles through P. Thus the curve V is of bidegree $(4, 2)$ and it decomposes as

$$V = F \times \{Q\} + F \times \{Q'\} + E \ ,$$

where E is a curve of bidegree $(2, 2)$. The branch points of the restriction $\pi_1|_E$ are the circles $S \in F$ touching C. Since C, C_1, C_2 are in general position, there are exactly four of them. So E is smooth elliptic by Lemma 7.2 below. The map t is just the composition of the covering involutions of $\pi_1|_E$ and $\pi_2|_E$, so it is a translation on E. The assumption implies that it is of order n. □

(7.2) Lemma. *Let $E \subset \mathbb{P}_1 \times \mathbb{P}_1$ a curve of bidegree $(2,2)$. Assume that the projections $\pi_1, \pi_2 : E \longrightarrow \mathbb{P}_2$ are finite and that one of them has at least four branch points. Then E is smooth.*

Proof. Suppose to the contrary that E is singular. First assume that E is reducible. Since the projections are finite, E is then a sum of two curves E_1, E_2 of bidegree $(1,1)$. By the adjunction formula $E_1 \cong E_2 \cong \mathbb{P}_1$. But then π_1 and π_2 have only two branch points, namely the intersection points of E_1 and E_2, a contradiction.

If E is irreducible, then it has exactly one singularity and the normalization $\widetilde{E}$ is a smooth rational curve. But then by the Hurwitz formula the map $\widetilde{E} \longrightarrow E \longrightarrow \mathbb{P}_1$ has only two branch points, a contradiction again. □

Next we turn to Steiner's theorem. In contrast to the situation of Emch's theorem here the relevant curve for the closing process is rational.

Let $C_1, C_2 \subset \mathbb{P}_2$ be smooth circles in general position and let F be a family of circles touching C_1 and C_2.

(7.3) Theorem (Steiner). *Suppose that there is a closed sequence of distinct circles $S_1, S_2, \ldots, S_n, S_{n+1} = S_1$ in F such that S_i touches S_{i+1} for $1 \le i \le n$. Then there are such sequences starting with any circle in F.*

Steiner's theorem is a consequence of the following

(7.4) Proposition. *a) $V := \{(S_1, S_2) \in F \times F \mid S_1 \in Q_{S_2}\}$ is a curve of bidegree $(4,4)$ and*

$$V = 2 \cdot \Delta + V_1 + r^* V_1 \ ,$$

where $\Delta \subset F \times F$ is the diagonal, V_1 is a curve of bidegree $(1,1)$ and $r : \mathbb{P}_1 \times \mathbb{P}_1 \longrightarrow \mathbb{P}_1 \times \mathbb{P}_1, (S_1, S_2) \longmapsto (S_2, S_1)$.

b) We have $V_1 \setminus \Delta \cong r^* V_1 \setminus \Delta \cong \mathbb{C}^*$. Let

$$t : V_1 \setminus \Delta \quad \longrightarrow \quad V_1 \setminus \Delta$$

be defined by $t(S_1, S_2) := (S_2, S_3)$, where $\pi_2^{-1}(S_2) = 2(S_2, S_2) + (S_1, S_2) + (S_3, S_2)$. Then t is the multiplication by a nonzero constant.

Proof. We may consider V as a subvariety of $\mathbb{P}_3 \times \mathbb{P}_3$, namely as the intersection of the varieties $F \times \mathbb{P}_3$, $\mathbb{P}_3 \times F$ and the hypersurface $\{(S_1, S_2) \mid S_1 \in Q_{S_2}\}$ of bidegree $(2,2)$. Denoting by $\pi_1, \pi_2 : V \longrightarrow F$ the projections, we have for $S_1 \in F$

$$\pi_2 \pi_1^{-1}(S_1) = Q_{S_1} \cap Q_{C_1}|_{\Pi(F)} \ ,$$

where $\Pi(F) \subset \mathbb{P}_3$ is the plane containing F. According to Lemma 6.4 the cones Q_{S_1}, Q_{C_1} touch along a line, hence the restriction to F of their intersection is of the form

$$Q_{S_1} \cap Q_{C_1}|_{\Pi(F)} = 2S_1 + S_2 + S_2' ,$$

where $S_2, S_2' \in F$. Therefore V contains the diagonal Δ as a component of multiplicity two, the residual curve V' being of bidegree $(2,2)$.

Now let S_0 be one of the null-circles associated to C_1 and C_2, i.e. S_0 consists of the two lines joining an intersection point P of C_1, C_2 with the infinitely far circle points. The quadric Q_{S_0} is then of rank 1, i.e. a double plane. It consists of the circles through P. The set $\pi_2\pi_1^{-1}(S_0)$ contains the circles through P touching C_1 and C_2, so $\pi_1^{-1}(S_0) = \{(S_0, S_0)\}$. We conclude that the two null-circles are branch points of both projections π_1, π_2. Because of $p_a(V') = 1$ this implies that V' is reducible. It consists of two curves V_1, V_2 of bidegree $(1,1)$ meeting in the points $(S_0, S_0), (S_0', S_0')$, where S_0, S_0' are the two null-circles associated to C_1 and C_2. Because of $r^*V = V$, we have $V_2 = r^*V_1$. □

8. The zig-zag theorem

In this section we give a proof of (a complex-projective version of) the zig-zag theorem [2] based on the consideration of an elliptic curve in the product of two circles.

For a point $P_0 = (x_0 : y_0 : z_0 : t_0)$ in $\mathbb{P}_3$ and a complex number r the quadric

$$S_{P_0,r} : (xt_0 - tx_0)^2 + (yt_0 - ty_0)^2 + (zt_0 - tz_0)^2 = r^2t^2t_0^2$$

is called the *sphere with center P_0 and radius r*. Its intersection with the infinitely far plane $t = 0$ is the conic $x^2 + y^2 + z^2 = 0$. A *circle* in $\mathbb{P}_3$ is a conic whose infinitely far points lie on this conic.

Now let $C_1, C_2 \subset \mathbb{P}_3$ be smooth circles and let a radius $r \in \mathbb{C}$ be fixed. A *zig-zag* for C_1, C_2 is a sequence of points $P_1, P_2, \dots$ such that for $i \geq 1$ the point P_i lies on C_1 resp. C_2, if i is odd resp. even, and such that $P_{i+1} \in S_{P_i,r}$. Consecutive points P_i, P_{i+1} in a zig-zag are thought of as having constant "distance" r. The zig-zag is said to *close after n steps*, if $P_{2n+1} = P_1$.

(8.1) Theorem (cf. [2]). *Let $C_1, C_2 \subset \mathbb{P}_3$ be a general pair of circles and let $r \in \mathbb{C}$. If the pair C_1, C_2 admits a zig-zag of distance r which closes after n steps, then it admits such zig-zags starting with any point on C_1.*

Proof. 1) We consider the curve

$$V := \{(P_1, P_2) \in C_1 \times C_2 \mid P_2 \in S_{P_1,r}\} .$$

It is the restriction of the hypersurface $\{P_2 \in S_{P_1,r}\} \subset \mathbb{P}_3 \times \mathbb{P}_3$ of bidegree $(2,2)$, so it is of bidegree $(4,4)$ in $C_1 \times C_2$. We denote by $\pi_i : V \longrightarrow C_i$, $i = 1, 2$, the projections and by Q_1, Q_1' resp. Q_2, Q_2' the infinitely far points on C_1 resp. C_2. For a point $P_1 \in C_1$, different from Q_1 and Q_1', we have

$$\pi_2\pi_1^{-1}(P_1) = S_{P_1,r} \cap C_2 = Q_2 + Q_2' + P_2 + P_2' \ ,$$

where $P_2, P_2' \in C_2$. Therefore V contains the lines

$$C_1 \times \{Q_2\},\ C_1 \times \{Q_2'\}$$

and by the same reasoning also the lines

$$\{Q_1\} \times C_2,\ \{Q_1'\} \times C_2 \ .$$

The residual curve E is of bidegree $(2, 2)$.

2) Next we determine the branch points of the projection $\pi_1|_E : E \longrightarrow C_1$ to show that E is smooth elliptic. Let us first consider the branch points of $\pi_1 : V \longrightarrow C_1$. These are the points $P_1 \in C_1$ such that the sphere $S_{P_1,r}$ touches C_2. We may assume C_2 to lie in the plane $z = 0$ having equation $x^2 + y^2 + t^2 = 0$, so $S_{P_1,r}$ intersects the plane of C_2 in the circle $S := S_{P_1,r}(x, y, 0, t)$. The sphere $S_{P_1,r}$ touches C_2 if and only if S lies on the tangent cone of C_2. Evaluating this condition by means of (6.2) we find

$$-\tfrac{1}{4}(x_1^2 + y_1^2 + z_1^2 - r^2t_1^2)^2 + \tfrac{1}{2}t_1^2(x_1^2 + y_1^2 + z_1^2 - r^2t_1^2) - x_1^2t_1^2 - y_1^2t_1^2 - \tfrac{1}{4}t_1^4 = 0 \ . \quad (10)$$

So the branch points of π_1 are just the intersection points of the circle C_1 and the quartic surface $Y \subset \mathbb{P}_3$ defined by (10). One may think of Y as the set of all points having "distance" r from C_2. Since Y intersects the plane $t_1 = 0$ in the double conic $(x_1^2 + y_1^2 + z_1^2)^2 = 0$, the points Q_1, Q_1' are both contained in $Y \cap C_1$ with multiplicity two. The four remaining points of intersection are the branch points of $\pi_1|_E : E \longrightarrow C_1$. For general C_1 these points are distinct, so E is in fact smooth elliptic.

3) Having identified the underlying elliptic curve, the proof now ends in the usual way: We denote by ι_1, ι_2 the covering involutions of $\pi_1|_E$ resp. $\pi_2|_E$ and by $t := \iota_2\iota_1$ the associated translation on E. The assumption that there is a zig-zag which closes after n steps implies that t is of order n. This proves the theorem. □

9. The Money-Coutts theorem

Let $C_1, C_2, C_3 \subset \mathbb{P}_2$ be smooth circles in general position. For each of the pairs C_1, C_2 and C_2, C_3 we choose one of the two families of circles touching both circles, say F_1, F_2 respectively. We are interested in the pairs of circles $S_1 \in F_1, S_2 \in F_2$ such that S_1 touches S_2. To begin with, we show:

(9.1) Proposition. *The curve*

$$V := \{(S_1, S_2) \in F_1 \times F_2 \mid S_1 \in Q_{S_2}\}$$

is of bidegree $(4, 4)$ *and we have* $V = 2\widetilde{\Delta} + E$*, where* $\widetilde{\Delta} \subset F_1 \times F_2$ *is a curve of bidegree* $(1, 1)$ *and* E *is a smooth elliptic curve.*

Proof. We may consider V as the subvariety

$$(F_1 \times \mathbb{P}_3) \cap (\mathbb{P}_3 \times F_2) \cap \{(S_1, S_2) \mid S_1 \in Q_{S_2}\}$$

of $\mathbb{P}_3 \times \mathbb{P}_3$. So for $S_1 \in F_1$ we have

$$\pi_2 \pi_1^{-1}(S_1) = Q_{S_1} \cap Q_{C_2}|_{\Pi(F_2)} \ ,$$

where $\Pi(F_2) \subset \mathbb{P}_3$ is the plane containing F_2 and $\pi_i : V \longrightarrow F_i$, $i = 1, 2$, are the projections. By Lemma 6.4 the cones Q_{S_1}, Q_{C_1} touch along a line, so their intersection restricts to $\Pi(F_2)$ as

$$Q_{S_1} \cap Q_{C_2}|_{\Pi(F_2)} = 2S_1' + S_2 + S_2' \ ,$$

where $S_1' \in F_2$ lies on the line of contact and $S_2, S_2' \in F_2$ lie on the residual conic. This shows that any line $\{S_1\} \times F_2 \subset F_1 \times F_2$ touches V, hence V must contain a curve $\widetilde{\Delta}$ of bidegree $(1, 1)$ as a component of multiplicity two. Next we consider the residual curve E, which is of bidegree $(2, 2)$ and we determine the branch points of the projection $\pi_1|_E : E \longrightarrow F_1$.

• First let S_1 be one of the two null-circles associated to C_1 and C_2. In this case the tangent quadric Q_{S_1} is a double plane. Therefore $\pi_2\pi_1^{-1}(S_1) = Q_{S_1} \cap Q_{C_2}|_{\Pi(F_2)}$ consists of two points only, so S_1 is a branch point of $\pi_1|_E$.

• Next let S_1 be an Apollonius circle of C_1, C_2, C_3, i.e. a circle touching all three of them. Further suppose that S_1 lies on F_1, but not on F_2. Since the tangent cones $Q_{C_1}, Q_{C_2}, Q_{C_3}$ are in general position, there are exactly two such circles. Since S_1 touches both C_2 and C_3, we have

$$\begin{aligned} Q_{S_1} \cap Q_{C_2} &= 2L_2 + D_2 \\ Q_{S_1} \cap Q_{C_3} &= 2L_3 + D_3 \ , \end{aligned}$$

where $L_2, L_3 \subset \mathbb{P}_3$ are lines and $D_2, D_3 \subset \mathbb{P}_3$ are conics. Thus set-theoretically

$$Q_{S_1} \cap Q_{C_2} \cap Q_{C_3} = L_2 \cap L_3 + D_2 \cap D_3 + L_2 \cap D_3 + L_3 \cap D_2$$

consists of five points at most. We claim that S_1 is a branch point of $\pi_1|_E$. To this end we show that only two of these points belong to F_2. We use the notation and the statement of Lemma 6.5: The conics D_2 and D_3 lie in planes $\Pi^{\varepsilon_{12}}_{S_1,C_2}$ resp. $\Pi^{\varepsilon_{13}}_{S_1,C_3}$, where $\varepsilon_{12}, \varepsilon_{13} \in \{\pm 1\}$, so $D_2 \cap D_3 \subset \Pi^{\varepsilon_{23}}_{C_2,C_3}$ for $\varepsilon_{23} := -\varepsilon_{12}\varepsilon_{13}$. Then L_2, L_3 lie in $\Pi^{-\varepsilon_{12}}_{S_1,C_2}$ resp. $\Pi^{-\varepsilon_{13}}_{S_1,C_3}$, so $L_2 \cap L_3 \subset \Pi^{\varepsilon_{23}}_{C_2,C_3}$ as well. Since $S_1 = L_2 \cap L_3$ does not belong to F_2 by assumption, we find

$$\begin{aligned} Q_{S_1} \cap Q_{C_2}|_{\Pi(F_2)} &= L_2 \cap D_3 + L_3 \cap D_2 \\ Q_{S_1} \cap Q_{C_3}|_{\Pi(F_2')} &= L_2 \cap L_3 + D_2 \cap D_3 \ , \end{aligned}$$

where F_2' is the second family of circles touching C_1 and C_2. This shows that $\pi_2\pi_1^{-1}(S_1)$ consists of the two points $L_2 \cap D_3, L_3 \cap D_2$.

Applying Lemma 7.2 we conclude that E is a smooth elliptic curve. $\square$

As before, let F_1, F_2 be families of circles touching C_1, C_2 resp. C_2, C_3, and suppose F_3 is a family of circles touching C_3 and C_1. By Proposition 9.1 we have three elliptic curves

$$\begin{array}{ccc} E_1 & \subset & F_1 \times F_2 \\ E_2 & \subset & F_2 \times F_3 \\ E_3 & \subset & F_3 \times F_1 \ . \end{array}$$

Let π_1, π_1' resp. π_2, π_2' resp. π_3, π_3' denote the projections onto the factors.

The next point we want to make is:

(9.2) Lemma. *Given F_1 and F_2, the family F_3 can be chosen in such a way that the projections π_1', π_2 as well as π_2', π_3 and π_3', π_1 have the same branch points in F_2 resp. F_3 resp. F_1.*

Proof. According to Lemma 6.5 the family F_3 can be chosen such that the planes $\Pi(F_1), \Pi(F_2), \Pi(F_3)$ have a line in common. The branch points of π_1' are the two null-circles of the pair C_1, C_2 and the two Apollonius circles in F_2, which do not belong to F_1. Because of our choice of F_3 these Apollonius circles do not belong to F_3 either. This shows that the branch points of π_1' coincide with those of π_2. The statement on the pairs π_2', π_3 and π_3', π_1 follows in the same way. $\square$

Now suppose that families F_1, F_2, F_3 have been chosen as above. The Money-Coutts theorem on the circles C_1, C_2, C_3 can then be stated as follows.

(9.3) Theorem (Tyrrel-Powell [11]). *Suppose that circles $S_1 \in F_1$, $S_2 \in F_2$, $S_3 \in F_3$ and $S_4 \in F_1$ are chosen such that $(S_1, S_2) \in E_1$, $(S_2, S_3) \in E_2$ and $(S_3, S_4) \in E_3$. Then it is possible to choose circles $S_5 \in F_2$, $S_6 \in F_3$ and $S_7 \in F_1$ in such a way that again $(S_4, S_5) \in E_1$, $(S_5, S_6) \in E_2$ and $(S_6, S_7) \in E_3$ and such that S_7 coincides with S_1.*

Proof. According to Lemma 9.2 the projections π_1', π_2 resp. π_2', π_3 resp. π_3', π_1 have the same branch points, hence there are isomorphisms $\varphi_1 : E_1 \longrightarrow E_2$, $\varphi_2 : E_2 \longrightarrow E_3$ and $\varphi_3 : E_3 \longrightarrow E_1$ such that $\pi_2 \circ \varphi_1 = \pi_1'$, $\pi_3 \circ \varphi_2 = \pi_2'$ and $\pi_1 \circ \varphi_3 = \pi_3'$. We identify $E_1 = E_2 = E_3 =: E$ and $\pi_1' = \pi_2, \pi_2' = \pi_3$ by means of φ_1 and φ_2. In this way φ_3 is identified with an automorphism t of E such that $\pi_1 \circ t = \pi_3'$. Since the elliptic curve E is determined by the intersection points of C_1, C_2 and two Apollonius circles of C_1, C_2, C_3, for general C_1, C_2, C_3 the curve E is general as well. Therefore the automorphism t is either a translation or an involution. By a suitable choice of φ_1 and φ_2 we can achieve that t is actually a translation. We denote by $\iota_1, \iota_2, \iota_3$ the covering involutions of π_1, π_2, π_3 on E.

We have $S_1 = \pi_1(e)$ for some $e \in E$. Then $S_2 \in \pi_1' \pi_1^{-1}(S_1)$, which means $S_2 = \pi_1'(\alpha_1 e)$ where $\alpha_1 \in \{1, \iota_1\}$. In the same way we proceed with S_3 and S_4 to get

$$S_4 = \pi_3'(\alpha_3 \alpha_2 \alpha_1 e) = \pi_1(t \alpha_3 \alpha_2 \alpha_1 e) \ ,$$

where $\alpha_i \in \{1, \iota_i\}$ for $1 \le i \le 3$. Now we determine the circles S_5, S_6, S_7 such that $(S_4, S_5), (S_5, S_6), (S_6, S_7) \in E$. We find

$$S_7 = \pi_3'(\beta_3\beta_2\beta_1 t\alpha_3\alpha_2\alpha_1 e) = \pi_1(t\beta_3\beta_2\beta_1 t\alpha_3\alpha_2\alpha_1 e) ,$$

where $\beta_i \in \{1, \iota_i\}$ for $1 \le i \le 3$. Now we make our choice for S_5, S_6, S_7 resp. for $\beta_1, \beta_2, \beta_3$ in the following way. We let $\beta_2 := \alpha_2$, $\beta_3 := \alpha_3$ and we choose β_1 such that the number of subscripts i, $1 \le i \le 3$, such that $\beta_i = \iota_i$ is odd. With these choices we obtain

$$t\beta_3\beta_2\beta_1 t\alpha_3\alpha_2\alpha_1 e = \beta_3\beta_2\beta_1\alpha_3\alpha_2\alpha_1 e = (\alpha_3\alpha_2\beta_1)^2\beta_1\alpha_1 e .$$

Since the composition of an odd number of involutions is again an involution, the latter expression equals $\beta_1\alpha_1 e$, so $S_7 = \pi_1(\beta_1\alpha_1 e) = \pi_1(e) = S_1$. □

Finally, we aim at a Poncelet-type statement in the situation of the Money-Coutts theorem. So suppose that C_1, C_2, C_3 are three circles as above and let families F_1, F_2, F_3 be chosen as before. As in the proof of the Money-Coutts theorem the elliptic curve defining the contact relation between circles of two families will be denoted by E.

Let $(S_i)_{i\ge1}$ be a sequence of circles such that $(S_i, S_{i+1}) \in E$ for $i \ge 1$ and

$$S_{3k+l} \in F_l \qquad \text{for } k \ge 0 \text{ and } 1 \le l \le 3 .$$

If it were to happen that $S_{3n+1} = S_1$ for some integer $n \ge 1$, then the sequence is said to *close after n steps.*

(9.4) Theorem. *Suppose there are more than 2^{3n+2} sequences $(S_i)_{i\ge1}$ closing after n steps. Then there are infinitely many sequences closing after n steps, starting with any given circle $S_1' \in F_1$.*

Proof. Let $S_1 = \pi_1(e)$, $e \in E$. The circles S_i, $i \ge 2$, are determined by repeatedly choosing automorphisms in $\{1, \iota_1, \iota_2, \iota_3\}$. So for $k \ge 1$ we have

$$\begin{aligned} S_{3k+1} &= \pi_1\left(\prod_{i=1}^k t\alpha_3^{(i)}\alpha_2^{(i)}\alpha_1^{(i)}\right) \\ S_{3k+2} &= \pi_1'\left(\alpha_1^{(k+1)}\prod_{i=1}^k t\alpha_3^{(i)}\alpha_2^{(i)}\alpha_1^{(i)}\right) \\ S_{3k+3} &= \pi_2'\left(\alpha_2^{(k+1)}\alpha_1^{(k+1)}\prod_{i=1}^k t\alpha_3^{(i)}\alpha_2^{(i)}\alpha_1^{(i)}\right) , \end{aligned}$$

where $t : E \longrightarrow E$ is a translation and $\alpha_j^{(i)} \in \{1, \iota_j\}$ for $1 \le j \le 3$ and $i \ge 1$. By assumption the sequence (S_i) closes after n steps, i.e. $S_{3n+1} = S_1$, i.e. $\pi_1(\gamma e) = \pi_1(e)$, where γ is the automorphism $\prod_{i=1}^n t\alpha_3^{(i)}\alpha_2^{(i)}\alpha_1^{(i)}$. The map γ is an involution or a translation on E, so replacing e by $\iota_1 e$ we may assume that γ is an involution. We have $\gamma(e) = e$ or $\gamma(e) = \iota_1(e)$. In the first case e is one of the four fixed points of γ.

Since there are 2^{3n} choices for the automorphisms $\alpha_j^{(i)}$, $1 \leq j \leq 3$, $1 \leq i \leq n$, this case occurs for at most 2^{3n+2} sequences. In the second case we conclude $\gamma = \iota_1$, so if we are given any circle $S_1' \in F_1$, $S_1' = \pi(e')$, then we define a sequence $(S_1')_{i \geq 1}$ by means of the automorphisms $\alpha_j^{(i)}$ and find

$$S_{3n+1} = \pi_1(\gamma(e')) = \pi_1(\iota_1(e')) = \pi_1(e') = S_1 \ .$$

□

References

[1] Barth, W., Michel, J.: Modular curves and Poncelet polygons. Math. Ann. 295, 25-49 (1993)

[2] Black, W.L., Howland, H.C., Howland, B.: A theorem about zig-zags between two circles. Am. Math. Monthly 81, 754-757 (1974)

[3] Emch, A.: An application of elliptic functions to Peaucellier's link-work (inversor). Ann. math., ser. 2 vol. 2, 60-63 (1901)

[4] Gerbaldi, F.: Le frazione continue di Halphen in relazione colle corrispondenze $[2,2]$ involutorie e coi poligoni di Poncelet. Rend. Circ. Mat. Palermo 43, 78-104 (1919)

[5] Griffiths, Ph., Harris, J.: A Poncelet theorem in space. Comment. Math. Helvetici 52, 145-160 (1977)

[6] Griffiths, Ph., Harris, J.: On Cayley's explicit solution to Poncelet's porism. L'enseignement math. 24, 31-40 (1978)

[7] Hitchin, N.: Poncelet polygons and the Painlevé equations. Warwick preprints 6/1994

[8] Hurwitz, A.: Ueber unendlich–vieldeutige geometrische Aufgaben, insbesondere aber die Schließungsprobleme. Math. Ann. 15, 8-15 (1879)

[9] Pedoe, D.: Circles. Pergamon Press, London New York Paris 1957

[10] Previato, E.: Poncelet's theorem and generalizations. Boston Univ seminar talk, Nov 1994

[11] Tyrrell, J.A., Powell, M.T.: A theorem in circle geometry. Bull. London Math. Soc. 3, 70-74 (1971)

[12] Weyr, E.: Über einige Sätze von Steiner und ihren Zusammenhang mit der zwei und zweigliedrigen Verwandtschaft der Grundgebilde ersten Grades. Crelles J. r. u. a. M. 71, 18-28 (1870)

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