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A universal measure for a pencil of conics and the Great Poncelet Theorem

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Abstract. Borel measures on conics which are invariant under the Poncelet map are investigated. For a pencil of conics the existence of a universal measure, which is invariant with respect to each conic in the pencil, is proved. Using this measure a new proof of the Great Poncelet Theorem is given. A full description of invariant Borel measures is also presented.

Bibliography: 10 titles.

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Introduction

Since Poncelet discovered (see [1]) one of the most beautiful theorems in geometry, on the closure of inscribed-circumscribed polygonal curves for two conics, many of its generalizations, consequences and applications have appeared in different fields of mathematics (see, for instance, [2]–[4] and the bibliographies in these papers). By the Great Poncelet Theorem (see [5]) one means a generalization of Poncelet's theorem to pencils of conics.

The idea of proving Poncelet's theorem using some invariant goes back to Jacobi and Bertrand (see [6]). In [7], King used a measure invariant under the Poncelet map[1] to give an extremely nice short proof of Poncelet's theorem, albeit only in the case of two disjoint ellipses, one inside the other. In the other cases of mutual position of the conics the order in which the vertices of the Poncelet polygonal line are arranged is disrupted and King's construction cannot be used. In [8] these difficulties were overcome using another construction, which involves homeoid density.

In this paper we extend King's construction of an invariant measure to a pencil of conics. It turns out that for such a pencil there exists a universal measure, which is invariant under the Poncelet map with respect to each conic in the pencil. We obtain an explicit description of this measure and use it for a new proof of the Great Poncelet Theorem, and also present a full description of all Borel measures on conics which are invariant under the Poncelet map.

First we recall several definitions. We shall deal with the projective plane obtained from the Euclidean plane by adding the line at infinity, which contains the points of intersection of parallel straight lines in the Euclidean plane. Points in the

[1]This assigns to a point on a conic the other point of its intersection with the tangent to another conic that passes through the point in question.

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Euclidean plane are called *proper* points of the projective plane, and points on the line at infinity are said to be *improper*. By a *conic* we mean a conical section, that is, the intersection of a plane and a circular cone. A (nondegenerate) conic can be an ellipse, a parabola or a hyperbola, depending on its mutual position with the line at infinity: an ellipse is disjoint from this line, a parabola is tangent to it, and a hyperbola intersects it at two points. All nondegenerate conics are projectively equivalent; apart from them there also exist degenerate conics: a pair of intersecting lines (the point of their intersection can lie at infinity), a pair of coinciding lines, and a point.

It is known that in a Cartesian (rectangular) system of coordinates the proper points of a conic satisfy an equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0,$$

in which at least one of the coefficients a, b and c is distinct from zero. We shall consider pairs $(\alpha, \mathcal{P}_\alpha)$, where α is a conic and $\mathcal{P}_\alpha = \mathcal{P}_\alpha(x, y)$ is a second-order polynomial defining α in the following sense:

$$\alpha = \bigl\{(x, y) \in \mathbb{R}^2 : \mathcal{P}_\alpha(x, y) = 0\bigr\}.$$

For each conic α there exist infinitely many polynomials $\mathcal{P}_\alpha$ with this property, but in the case of a nondegenerate conic or a conic which degenerates into a pair of straight lines all of them are proportional.

By definition a *pencil of conics* $\mathcal{F}(\alpha, \beta)$ generated by two nondegenerate conics α and β is the one-parameter family of conics defined by the equations

$$\lambda \mathcal{P}_\alpha(x) + \mathcal{P}_\beta(x) = 0, \quad \text{where} \quad \lambda \in \mathbb{R} \cup \infty.$$

The conics α and β themselves correspond to $\lambda = \infty$ and $\lambda = 0$, respectively, so they also belong to $\mathcal{F}(\alpha, \beta)$.

By *Poncelet trajectories* of the conics α and β we shall mean the sequences of vertices of inscribed/circumscribed polygonal lines. Poncelet's theorem states that either all the Poncelet trajectories return to the starting position after the same number of steps or no Poncelet trajectory returns.

In the Great Poncelet Theorem, in place of two conics α and β we take several conics $\alpha, \beta_1, \dots, \beta_n$ in some pencil (where α can also be a degenerate conic, a pair of lines). Next we construct a polygonal line inscribed in α such that its sides are tangent to $\beta_1, \dots, \beta_n$ in turn. Then the result can be a closed polygonal line with n segments. On the other hand, under certain conditions on tangencies with $\beta_1, \dots, \beta_n$ there exist infinitely many such closed inscribed/circumscribed polygonal lines and any point in α from which a tangent to β_1 can be drawn can be the first vertex of such a polygonal line.

By a polygon circumscribed about $\beta_1, \dots, \beta_n$ we should mean more than just a polygon whose sides are tangent to the conics $\beta_1, \dots, \beta_n$ in turn: this condition alone is not sufficient for the theorem to be true. We define a circumscribed polygon as follows. If α is a nondegenerate conic, then by its *exterior region* we shall mean the set of points through which at least one tangent to α can be drawn. If α is a degenerate conic equal to a pair of lines, then we mean by its exterior region the interior of one of the vertical corners formed by these lines. We say that a side of a polygon makes an *exterior* tangency with a conic β_i if the tangency occurs at a point in the exterior region of α. Now we say that an n-gon inscribed in α such that its ith side is tangent to β_i is *circumscribed* about $\beta_1, \dots, \beta_n$ if the number of exterior tangencies is even.

In his treatise *Les coniques* [9] Lebesgue stated the Great Poncelet Theorem using another definition of a circumscribed polygon, but on a real plane these two definitions are equivalent, and the above definition is more convenient for us.

Theorem (Lebesgue's statement of the Great Poncelet Theorem). *Let $\alpha, \beta_1, \dots, \beta_n$ be conics in some pencil. If there exists a polygon inscribed in α and circumscribed about $\beta_1, \dots, \beta_n$, then there exist infinitely many such polygons.*

To specify such a polygon the following data can be taken arbitrarily:

(i) *the order in which its sides are tangent to $\beta_1, \dots, \beta_n$ (for instance, $\beta_{\sigma(1)}, \dots, \beta_{\sigma(n)}$, where σ is a permutation of $\{1, \dots, n\}$)*;

(ii) *the tangent to $\beta_{\sigma(1)}$ which contains a side of the polygon*;

(iii) *the point of intersection of this tangent with α that belongs to the side tangent to $\beta_{\sigma(2)}$*.

We give the proof of this theorem in § 3. First we define the Poncelet map on a conic and construct an invariant measure for it. Then we prove the universality of this measure, that is, its invariance under the Poncelet map with respect to each conic in the pencil. Next we construct a so-called equalizing map, which reduces the Poncelet map to a rotation of a circle, and the results in § 2 will imply a geometric description of the rotation angle for each conic in the pencil. Separately, in § 4 we treat the degenerate case, when the conic α is a pair of lines, and the conics in the pencil are disjoint. Finally, in § 5 we present a classification of all invariant measures for conics and all universal measures for pencils of conics. For instance, we show there that the universal measure is always unique, while the question of whether an invariant measure is unique hinges on whether Poncelet trajectories are closed or not.

§ 1. Invariant measure

In this section we define the Poncelet map on a conic, construct a measure invariant under it and introduce the notion of a universal measure of a pencil.

Two distinct conics α and β can have from 0 to 4 common points, which are also common to all conics in the pencil $\mathcal{F}(\alpha, \beta)$. A common point is a point of intersection or a point of tangency of the conics. We shall treat a point of tangency as 2 or 3 merged points of intersection (depending on the multiplicity of tangency). Taking this convention there can be 0, 2 or 4 common points, some of which can 'go to infinity' in the case when the asymptotic directions coincide.

We look at the set α_β of points in α such that a tangent line to β can be drawn through each of them. For each nondegenerate conic γ in $\mathcal{F}(\alpha, \beta)$ the set α_γ either coincides with α_β or coincides with the complement $\alpha \setminus \alpha_\beta$. If for conics $\beta_1, \dots, \beta_n$ in $\mathcal{F}$ a polygon inscribed in α and circumscribed about $\beta_1, \dots, \beta_n$ exists, then the set α_{β_i} is the same for all $i = 1, \dots, n$. Hence in discussing the system of conics $\alpha, \beta_1, \dots, \beta_n$ in the Great Poncelet Theorem we denote this set by $\alpha_{\mathcal{F}}$.

As a point x moves along α_β, each of the two tangents from x to β varies in a continuous fashion and defines a certain family of straight lines. Thus, for each $x \in \alpha_\beta$ there is a unique tangent $l_1(x)$ from one family and a unique tangent $l_2(x)$ from the other. For $k = 1, 2$ we define the map $j^k_\beta\colon \alpha_\beta \to \alpha_\beta$ that associates with each $x \in \alpha_\beta$ the second point of intersection of the line $l_k(x)$ with the conic α.

We note straight away that j^1_β and j^2_β are continuous maps, and if the conics α and β are disjoint, then both j^1_β and j^2_β are bijective. To make them bijective in the case of intersecting conics as well we proceed as follows: taking two copies α^1_β and α^2_β of the set α_β, we denote their union by $\widetilde{\alpha}_\beta$. Now we extend j^1_β and j^2_β to $\widetilde{\alpha}_\beta$ as follows. If the tangency of $l_k(x)$ with β is interior for α, then let $j^k_\beta(x)$ lie in the same copy of α_β as x, and if this is an exterior tangency, then let it lie in the other copy. For example, the image of the arc UV in Fig. 1 under one of the maps j^1_β, j^2_β is the arc WBW (that is, WB taken twice).

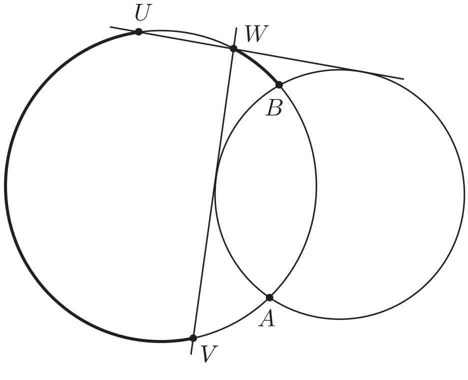

Figure 1

That is, we only go over from one copy of α_β to the other when we have exterior tangency.

Now the maps j^1_β and j^2_β are both bijective and, furthermore, mutually inverse.

Definition 1.1. We call the maps j^1_β and j^2_β the *Poncelet maps with respect to the conic* β.

The arcs on α^1_β and on α^2_β generate the Borel σ-algebra $\mathscr{B}(\widetilde{\alpha}_\beta)$ on $\widetilde{\alpha}_\beta$.

Definition 1.2. We say that a measure μ on $\mathscr{B}(\widetilde{\alpha}_\beta)$ is j_β*-invariant* if

$$\mu(j^1_\beta(X)) = \mu(X) = \mu(j^2_\beta(X)) \quad \text{for all} \quad X \in \mathscr{B}(\widetilde{\alpha}_\beta).$$

That is, a measure is said to be invariant if it is invariant under the Poncelet map. For the construction of an invariant measure we shall require the following simple result.

Lemma 1.3. *Assume that the tangents to a conic α at points A and B intersect at a point P. Then*

$$\frac{PA}{|\nabla \mathcal{P}_\alpha(A)|} = \frac{PB}{|\nabla \mathcal{P}_\alpha(B)|}, \tag{1.1}$$

where

$$|\nabla \mathcal{P}_\alpha(x,y)| = \left|\left\{\frac{\partial \mathcal{P}_\alpha}{\partial x}, \frac{\partial \mathcal{P}_\alpha}{\partial y}\right\}\right| = \sqrt{\left(\frac{\partial \mathcal{P}_\alpha}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{P}_\alpha}{\partial y}\right)^2}.$$

Proof. Since the length of the gradient vector keeps its value after an orthogonal change of coordinates we can verify the claim in any Cartesian coordinate system.

1) Let α be a parabola. We choose a coordinate system in which α has equation

$$y = ax^2, \qquad a \in \mathbb{R}.$$

Then

$$|\nabla \mathcal{P}_\alpha(x,y)| = \sqrt{1 + 4a^2x^2}.$$

The tangents to α at points $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by

$$y = ax_1(2x - x_1), \qquad y = ax_2(2x - x_2).$$

Their intersection point P has the coordinates

$$\left(\frac{x_1 + x_2}{2}, ax_1x_2\right).$$

Then

$$|PA| = \frac{1}{2}|x_2 - x_1|\sqrt{1 + 4a^2x_1^2}, \qquad |PB| = \frac{1}{2}|x_2 - x_1|\sqrt{1 + 4a^2x_2^2}.$$

2) Let α be an ellipse or a hyperbola. Then we take a coordinate system in which α has equation

$$ax^2 + by^2 = 1, \qquad a, b \in \mathbb{R}.$$

Then

$$|\nabla \mathcal{P}_\alpha(x,y)| = 2\sqrt{a^2x^2 + b^2y^2}.$$

The tangents to α at points $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by

$$ax_1x + by_1y = 1, \qquad ax_2x + by_2y = 1.$$

Their intersection point P has the coordinates

$$\left(\frac{y_2 - y_1}{ax_1y_2 - ax_2y_1}, \frac{x_1 - x_2}{bx_1y_2 - bx_2y_1}\right).$$

Then

$$|PA| = \frac{1 - ax_1x_2 - by_1y_2}{ab|x_1y_2 - x_2y_1|}\sqrt{a^2x_1^2 + b^2y_1^2},$$
$$|PB| = \frac{1 - ax_1x_2 - by_1y_2}{ab|x_1y_2 - x_2y_1|}\sqrt{a^2x_2^2 + b^2y_2^2}.$$

It is easy to see that (1.1) holds in either case.

Remark 1.4. The lemma also holds for a degenerate quadric equal to a pair of lines (then the point P will always be the vertex of the conic). This holds by reasons of continuity because a degenerate conic is the limit of nondegenerate ones.

Theorem 1.5. *Let α and β be two distinct (perhaps degenerate) conics. Then the function*

$$\mu_\beta(\mathcal{A}) = \int_{\mathcal{A}} \frac{dl}{|\nabla \mathcal{P}_\alpha(x,y)|\sqrt{\mathcal{P}_\beta(x,y)}}, \qquad \mathcal{A} \in \mathcal{B}(\widetilde{\alpha}_\beta), \tag{1.2}$$

where dl is the differential of arc length on α, defines a j_β-invariant measure on α.

Note first that

$$f(X) = \frac{1}{|\nabla \mathcal{P}_\alpha(X)|\sqrt{\mathcal{P}_\beta(X)}}$$

tends to infinity as X approaches a point in β. However, if α is a nondegenerate conic, then $f(X)$ is bounded outside any neighbourhood of β and therefore is integrable on any smooth curve (which is not necessarily bounded because $f(X)$ has an integrable singularity at infinity). On the other hand, if a smooth curve L has common points with β then the integral of $f(X)$ over L can be treated as an improper one. It converges if and only if L is nowhere tangent to β. Indeed, in a neighbourhood of a common point X_0 of L and β the function $f(X)$ has a singularity. If the conic and the curve intersect at X_0, then this is an integrable singularity of type $O(1/\sqrt{t})$ as $t \to 0$, but if they are tangent, then this is a nonintegrable singularity of type $O(1/t)$ as $t \to 0$.

Hence the measure μ_β is well defined for intersecting conics, while if α and β are tangent, then we can consider this measure as defined outside an arbitrarily small neighbourhood of the point of tangency.

If α degenerates into a pair of lines, then $f(X)$ has a nonintegrable singularity at the vertex P of α. In this case the measure μ is infinite, but the measure of any sets whose closure does not contain P is finite.

Proof of Theorem 1.5. Let $B = j_\beta(A)$, $B' = j_\beta(A')$, let C be the point of tangency of the line AB and the conic β, and let $D = AB \cap A'B'$ (Fig. 2).

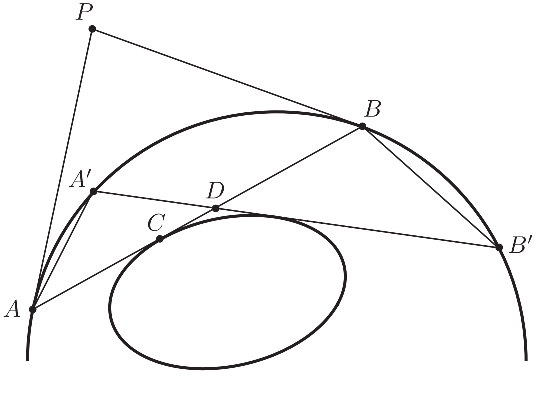

Figure 2

By the sine theorem for the triangles $AA'D$ and $B'BD$

$$\frac{AA'}{\sin(\angle A'DA)} = \frac{AD}{\sin(\angle AA'D)}, \qquad \frac{BB'}{\sin(\angle BDB')} = \frac{BD}{\sin(\angle BB'D)}.$$

Dividing the first equality by the second we find

$$\frac{AA'}{BB'} = \frac{AD}{BD}\,\frac{\sin(\angle BB'D)}{\sin(\angle AA'D)}. \tag{1.3}$$

Let P denote the point of intersection of the tangents to α at A and B. Taking the limit as $A' \to A$ equality (1.3) yields

$$\frac{dA}{dB} = \frac{AC}{BC}\,\frac{\sin(\angle PBA)}{\sin(\angle PAB)},$$

where dX is the differential of arc length on α at the point X. Now the sine theorem for the triangle PAB yields

$$\frac{dA}{dB} = \frac{AC}{BC}\,\frac{PA}{PB}. \tag{1.4}$$

The function $\mathcal{P}_\beta$ on the line AB is quadratic. Since the quadratic form $\mathcal{P}_\beta$ keeps sign outside the conic β and vanishes at C, for points X on AB we have

$$\mathcal{P}_\beta(X) = k|CX|^2$$

with some constant k. But then

$$\frac{AC}{BC} = \frac{\sqrt{\mathcal{P}_\beta(A)}}{\sqrt{\mathcal{P}_\beta(B)}},$$

and, in view of Lemma 1.3, equality (1.4) yields

$$\frac{dA}{dB} = \frac{\sqrt{\mathcal{P}_\beta(A)}}{\sqrt{\mathcal{P}_\beta(B)}}\,\frac{|\nabla\mathcal{P}_\alpha(A)|}{|\nabla\mathcal{P}_\alpha(B)|},$$

or

$$\frac{dA}{|\nabla\mathcal{P}_\alpha(A)|\sqrt{\mathcal{P}_\beta(A)}} = \frac{dB}{|\nabla\mathcal{P}_\alpha(B)|\sqrt{\mathcal{P}_\beta(B)}}. \tag{1.5}$$

Now it follows from (1.5) that $\mu_\beta(j_\beta(\mathcal{A})) = \mu_\beta(\mathcal{A})$.

We see that measures invariant under the Poncelet maps do exist. For each conic γ in the pencil $\mathcal{F}$ we take the measure μ_γ.

Definition 1.6. We say that a Borel measure on α is $\mathcal{F}$-*universal* if it is invariant under the Poncelet maps with respect to all the conics in $\mathcal{F}$.

It turns out that for all $\gamma \in \mathcal{F}$ the measure μ_γ is the same, that is, we have the following result.

Theorem 1.7. *The measure defined in Theorem 1.5 is $\mathcal{F}$-universal.*

Proof. We look at an arbitrary conic γ in $\mathcal{F}$. There exists λ such that $\mathcal{P}_\gamma = \lambda\mathcal{P}_\alpha + \mathcal{P}_\beta$. Since $\mathcal{P}_\alpha(X) = 0$ for each point X in α, it follows that $\mathcal{P}_\gamma(X) = \mathcal{P}_\beta(X)$. Hence $\mu_\gamma(\mathcal{A}) = \mu_\beta(\mathcal{A})$ for each set $\mathcal{A} \in \mathcal{B}(\widetilde{\alpha}_\mathcal{F})$, that is, for any conics β and γ in $\mathcal{F}$ the corresponding measures μ_β and μ_γ are the same.

Thus we do not need the subscript in the notation μ_β and can denote this measure by μ.

§ 2. Some geometric properties of pencils of conics and of the $\mathcal{F}$-universal measure

The results in this section will enable us in what follows to give a geometric description of the absolute values of the vectors of the translations corresponding to the Poncelet maps after the equating mapping. We recall several definitions from projective geometry.

Let α be a nondegenerate conic. Consider straight lines through some point P. At the points of intersection with α of each of these lines we draw tangents to α. The locus of points of intersection of these tangents is a straight line p, which is called the *polar* of P with respect to α. The point P is called the *pole* of the line p. The pole of a straight line is the point of intersection of the polars of points on this line.

If each side of a triangle is the polar of the opposite vertex, then the triangle is said to be *self-polar* with respect to the conic α.

A mapping i defined on a conic is a projective involution if and only if for each point X on the conic the lines $Xi(X)$ intersect at one point. This point is called the *Frégier point* of the involution i.

A detailed presentation of the geometric properties of these objects can be found in [5] and [10].

There exist at most 3 critical values of the parameter λ for which $\gamma_\lambda = \lambda\alpha + \beta$ is a degenerate conic. These are the real roots of the cubic equation $\det(A_\lambda) = 0$, where A_λ is the matrix of γ_λ. The conic γ_λ degenerates into a point or a pair of straight lines.

By the *vertex* of the degenerate conic we mean the point into which the conic has degenerated or the point of intersection of a pair of lines if the conic has degenerated into these, or an arbitrary point on the conic if it has degenerated into a pair of coinciding lines. A tangent to a degenerate conic is a straight line through its vertex, so that the Poncelet maps j^1_δ and j^2_δ with respect to a degenerate conic δ coincide with the involution of α whose Frégier point is the vertex of δ. The measure μ is also j_δ-invariant.

Proposition 2.1. *Suppose that conics α and β intersect at points A, B, C and D and that the tangents to β at these points intersect α also at points a, b, c and d, respectively* (*Fig.* 3). *Then $\mu(Ab) = \mu(Ba) = \mu(Cd) = \mu(Dc)$.*

Proof. We claim that the lines AC, BD, ac and bd intersect at one point. For the proof we look at the projective transformation taking the quadrilateral $ABCD$ into a parallelogram. Then $P = AC \cap BD$ goes to the centre of the images of α and β, and the required result is obvious by the symmetry in the image of the point P.

The equalities

$$\mu(Ab) = \mu(Ba), \qquad \mu(Cd) = \mu(Dc)$$

follow from the j_β-invariance of μ, and the equality $\mu(Ab) = \mu(Cd)$ follows from its j_δ-invariance, where δ is the degenerate conic $AC \cup BD$ in the pencil $\mathcal{F}(\alpha, \beta)$.

Proposition 2.2. *Let α and β be disjoint conics with four common tangents Aa, Bb, Cc and Dd, where A, B, C and D are points on α and a, b, c and d are points on β* (*Fig.* 4). *Then*

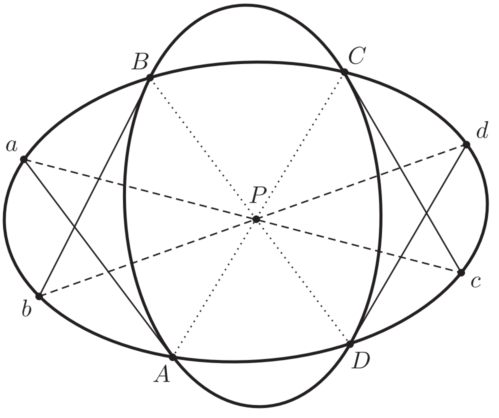

Figure 3

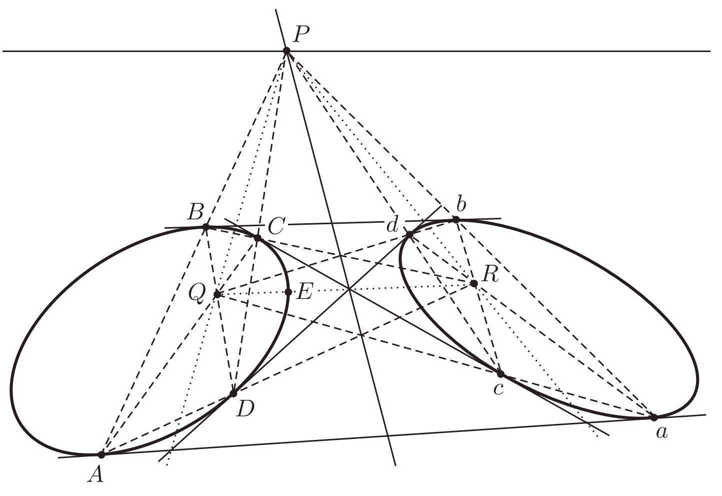

Figure 4

a) $P := AB \cap CD = ab \cap cd$, $Q := AC \cap BD = ac \cap bd$, $R := AD \cap BC = ad \cap bc$, *so that the quadrilaterals $ABCD$ and $abcd$ have the same diagonal triangle PQR*;

b) *the triangle PQR is self-polar with respect to any conic in the pencil $\mathcal{F}(\alpha, \beta)$*;

c) *the points P, Q and R are the vertices of three degenerate conics in the pencil*;

d) $\mu(BC) = \mu(AD)$, $\mu(AB) = \mu(CD)$, $\mu(CE) = \mu(DE)$, *where E is a point of intersection of the straight line QR and α.*

Proof. Since the line PQ is the polar of R with respect to α, it passes through the points of intersection of the tangents Bb and Cc and the tangents Aa and Dd. Now it follows from the duality principle that the pole of PQ with respect to β is the point of intersection of the lines bc and ad. For the lines PR and QR similar arguments show that PQR is also the diagonal triangle of the quadrilateral $abcd$. Hence the triangle PQR is self-polar with respect to both α and β, and therefore it is self-polar with respect to any conic in $\mathcal{F}(\alpha, \beta)$.

Since conics in the pencil are disjoint, there is a unique conic through every point in the plane. We look at the conic δ_P through the point P. Were it nondegenerate, the polar of P with respect to δ_P would be equal to the line QR, but on the other hand it would be tangent to δ_P at P. Then P, Q and R would be collinear, which is impossible since PQR is a self-polar triangle. Hence δ_P is a degenerate conic. In a similar way the conics δ_Q and δ_R through Q and R are also degenerate.

If a triangle is self-polar with respect to a conic, then one of its vertices must lie inside the conic[2], and the other two vertices must lie outside it. For definiteness we assume that Q lies inside α, while R lies inside β.

If the conic δ_Q contains a straight line, then this line intersects α because Q lies inside α. Then all the conics in $\mathcal{F}(\alpha, \beta)$ pass through this point of intersection. However, α and β are disjoint, and therefore δ_Q coincides with the point Q. In a similar way, the conic δ_R degenerates into the point R. Now it follows from the j_{δ_Q}-invariance of μ that $\mu(BC) = \mu(AD)$ and $\mu(AB) = \mu(CD)$. Since μ is also j_{δ_R}-invariant and the composite map $j_{\delta_Q} \circ j_{\delta_R}$ takes the arc CE to the arc DE, we have $\mu(CE) = \mu(DE)$.

Lemma 2.3. *Assume that the conics in the pencil $\mathcal{F}$ are disjoint, let P, Q and R be the vertices of the degenerate conics in this pencil and let α and β be nondegenerate conics in $\mathcal{F}$ (Fig. 5). Consider two straight lines through P, the first of which intersects α at points M and N, while the second intersects β at K and L. Then the straight lines MK, NL and QR intersect at one point.*

Proof. Let S denote the point of intersection of the lines MN and QR and T denote the point of intersection of the lines KL and QR. Since QR is the polar of P with respect to both α and β, the point S is the harmonic conjugate of P with respect to M and N, and T is the harmonic conjugate of P with respect to K and L. Hence the line QR is the polar of P with respect to the degenerate conic equal to the pair of lines MK and NL, and therefore it passes through the point of intersection of these lines.

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[2]For a nondegenerate conic, by its interior we mean the set of points in the plane that have the following property: any straight line through the point under consideration meets the conic.

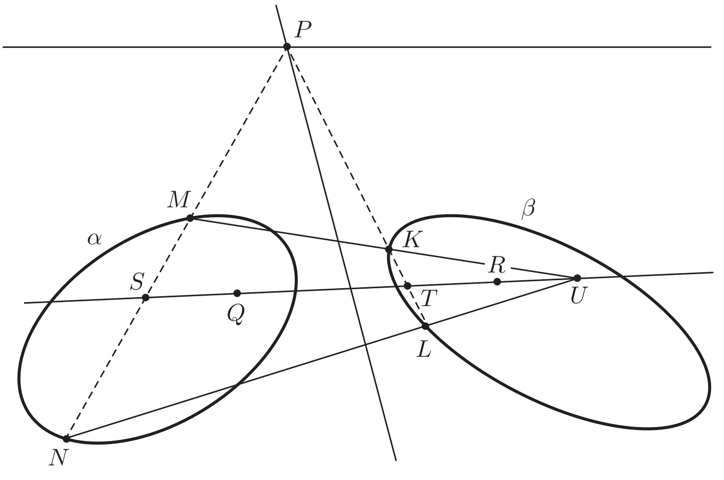

Figure 5

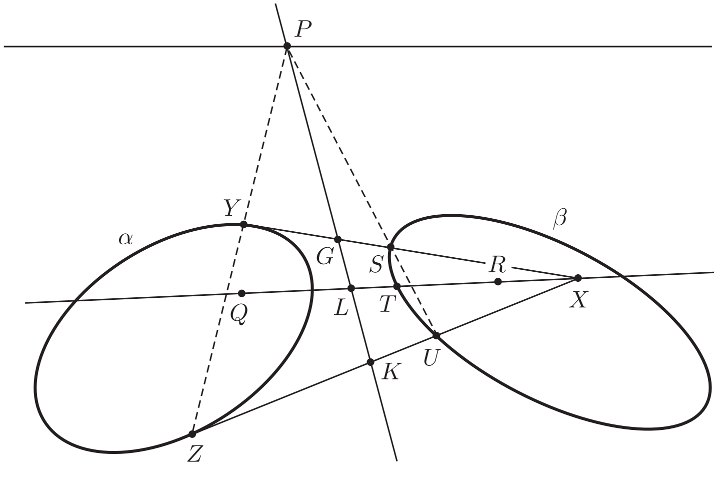

Figure 6

So far μ has denoted the measure on the conic α defined in Theorem 1.5. Note that a similar measure can be defined on each conic γ in the pencil $\mathcal{F}$. To avoid confusion, in what follows we shall denote this measure by μ^γ.

Proposition 2.4. *Suppose that conics in $\mathcal{F}$ are disjoint, let P be the vertex of a degenerate conic δ_P in $\mathcal{F}$ equal to a pair of lines, let Q and R be the vertices of the other two degenerate conics in $\mathcal{F}$, and let α and β be nondegenerate conics in $\mathcal{F}$ (Fig. 6). For an arbitrary point X on the line QR which lies inside β let XY and XZ be the rays tangent to α, which intersect β at points S and U, respectively, and which intersect one of the lines forming the degenerate conic δ_P at points G and K such that the interval GK does not contain P. Let T and L be points of intersection of the ray XQ with the conic β and the line GK, respectively. Then*

a) $\mu^\beta(ST) = \mu^\beta(TU)$;

b) $\mu^{\delta_P}(GL) = \mu^{\delta_P}(LK)$.

Proof. a) Since X lies on the polar QR of the point P with respect to α, P lies on the polar YZ of X. Then by Lemma 2.3 the straight line SU also contains P. However, the tangent to β at T also passes through P. Hence the Poncelet map j_{δ_P} of the conic β takes the arc ST to the arc TU and $\mu^\beta(ST) = \mu^\beta(TU)$ by the j_{δ_P}-invariance of μ^β.

b) Now let β be a variable conic: $\beta = \beta_\lambda = \lambda\alpha + \delta_P$ for $\lambda \in \mathbb{R}$. Let

$$f_\lambda := \frac{1}{|\nabla \mathcal{P}_{\beta_\lambda}| \sqrt{\mathcal{P}_\alpha}}$$

be the density of the measure μ^{β_λ}. By what we proved in a),

$$\int_{ST} f_\lambda \, dl = \int_{TU} f_\lambda \, dl. \tag{2.1}$$

Fix $\varepsilon > 0$. As $\lambda \to 0$, the conic β_λ tends to δ_P. Since for each $\lambda \in \mathbb{R}$ the function f_λ is continuous in a sufficiently small neighbourhood V of the line segment GK, there exists $\lambda_0 > 0$ such that

$$\biggl|\int_{GL} f_\lambda \, dl - \int_{ST} f_\lambda \, dl\biggr| < \frac{\varepsilon}{4}, \qquad \biggl|\int_{LK} f_\lambda \, dl - \int_{TU} f_\lambda \, dl\biggr| < \frac{\varepsilon}{4} \tag{2.2}$$

for $\lambda < \lambda_0$.

At each point in V, for sufficiently small λ we have

$$|f_\lambda - f_0| < \frac{\varepsilon}{4|GK|},$$

so that

$$\biggl|\int_{GL} f_\lambda \, dl - \int_{GL} f_0 \, dl\biggr| < \frac{\varepsilon}{4}, \qquad \biggl|\int_{TU} f_\lambda \, dl - \int_{TU} f_0 \, dl\biggr| < \frac{\varepsilon}{4}. \tag{2.3}$$

Then

$$\begin{aligned}\left|\int_{GL} f_0\,dl - \int_{LK} f_0\,dl\right| &< \left|\int_{GL} f_0\,dl - \int_{GL} f_\lambda\,dl\right| + \left|\int_{GL} f_\lambda\,dl - \int_{ST} f_\lambda\,dl\right| \\ &\quad + \left|\int_{ST} f_\lambda\,dl - \int_{TU} f_\lambda\,dl\right| + \left|\int_{TU} f_\lambda\,dl - \int_{TU} f_0\,dl\right| + \left|\int_{TU} f_0\,dl - \int_{LK} f_0\,dl\right| \\ &\overset{(2.1)\text{–}(2.3)}{<} \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

Since this holds for any $\varepsilon > 0$, it follows that

$$\int_{GL} f_0\,dl = \int_{LK} f_0\,dl,$$

that is, $\mu^{\delta_P}(GL) = \mu^{\delta_P}(LK)$.

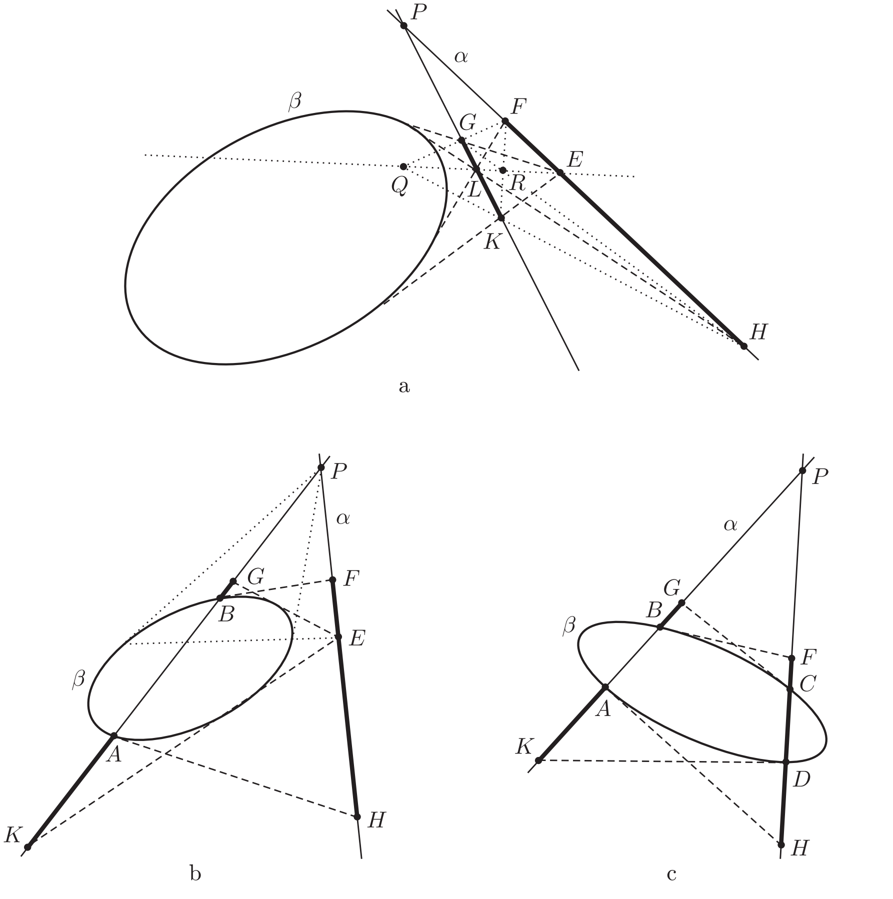

Figure 7

Proposition 2.5. *Let α be a degenerate conic equal to a pair of lines and β be some nondegenerate conic. Then the following relations hold for the measure $\mu = \mu^{\alpha}$, depending on the mutual position of α and β:*

a) $\mu(LK) = \mu(LG) = \mu(EF) = \mu(EH)$ (*Fig.* 7, a);
b) $\mu(AK) = \mu(BG) = \mu(EF) = \mu(EH)$ (*Fig.* 7, b);
c) $\mu(AK) = \mu(BG) = \mu(CF) = \mu(DH)$ (*Fig.* 7, c).

Relation a) is a consequence of Proposition 2.4 and the j_{δ_Q}- and j_{δ_R}-invariance of μ; relations b) and c) are proved in a similar way.

§ 3. The equating mapping. The proof of the Great Poncelet Theorem

The key point in the proof of the Great Poncelet Theorem is the use of the equating mapping which transforms the Poncelet maps with respect to all conics in the pencil into translations along a circle. In what follows we define this mapping, establish the equating property and then prove the Great Poncelet Theorem. Our arguments below work well also in the case when α is a degenerate conic, equal to a pair of straight lines with vertex lying inside the conics in $\mathcal{F}$. We discuss the case when α is a pair of lines whose vertex lies outside the conics in $\mathcal{F}$ separately in § 4 (in this case the measure is infinite and cannot be normalized).

Throughout this section we assume that

1) the measure μ is normalized as follows:

$$\mu(\alpha_\beta) = \begin{cases} \dfrac{1}{2} & \text{if } \{\alpha \cap \beta\} \text{ consists of two points with odd multiplicity,} \\ 1 & \text{otherwise;} \end{cases}$$

2) $\mu(\alpha \setminus \alpha_\beta) = 0$;

3) for any points $X, Y \in \widetilde{\alpha}_{\mathcal{F}}$ we mean by (X, Y) the arc XY of α such that the motion from X to Y is anticlockwise; then for any three points $X, Y, Z \in \widetilde{\alpha}_{\mathcal{F}}$,

$$\mu(X, Y) + \mu(Y, Z) \equiv \mu(X, Z) \pmod 1.$$

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be a circle with operation $\oplus$ of addition modulo 1. Now we define the *equating mapping* $\vartheta\colon \widetilde{\alpha}_{\mathcal{F}} \to \mathbb{T}$. Let X be an arbitrary point in $\widetilde{\alpha}_{\mathcal{F}}$ such that X belongs to the copy $\alpha^i_{\mathcal{F}}$ of $\alpha_{\mathcal{F}}$. Then we set

$$\vartheta\colon X \mapsto (-1)^i \mu(E, X), \tag{3.1}$$

where E is the point in α defined in Proposition 2.1 in the case of disjoint conics in $\mathcal{F}$, while in the case of intersecting conics in $\mathcal{F}$, E coincides with some point of intersection.

Thus, X as a point in $\alpha^1_{\mathcal{F}}$ and X as a point in $\alpha^2_{\mathcal{F}}$ are taken to two points in $\mathbb{T}$ symmetric with respect to the origin. Furthermore, if the mutual position of the conics ensures the equality $\mu(\alpha_\beta) = 1/2$, then ϑ is injective on the whole of $\widetilde{\alpha}_{\mathcal{F}}$. And when the measure μ is normalized so that its total mass is 1, ϑ is injective on both $\alpha^1_{\mathcal{F}}$ and $\alpha^2_{\mathcal{F}}$.

Let $\rho_c\colon \mathbb{T} \to \mathbb{T}$ denote translation by c, so that $\rho_c\colon x \mapsto x \oplus c$.

Proposition 3.1. *Each conic β in $\mathcal{F}$ is associated with a number c_β such that the equating mapping transforms the Poncelet maps j_β^1 and j_β^2 into translations by $\pm c_\beta$. More precisely, for each $k \in \{1,2\}$ the diagram*

$$\begin{array}{ccc} \widetilde{\alpha}_{\mathcal{F}} & \xrightarrow{\ j_\beta^k\ } & \widetilde{\alpha}_{\mathcal{F}} \\ \big\downarrow{\scriptstyle \vartheta} & & \big\downarrow{\scriptstyle \vartheta} \\ \mathbb{T} & \xrightarrow{\ \rho_{(-1)^k c_\beta}\ } & \mathbb{T} \end{array}$$

is commutative:

$$\vartheta \circ j_\beta^k = \rho_{(-1)^k c_\beta} \circ \vartheta. \tag{3.2}$$

Proof. If the line $Zj_\beta^k(Z)$ and β make an interior tangency with respect to α, then the points Z and $j_\beta^k(Z)$ move in the same direction on α and lie in the same copy $\alpha_{\mathcal{F}}^i$ of $\alpha_{\mathcal{F}}$. If $Zj_\beta^k(Z)$ and β make an exterior tangency with respect to α, then Z and $j_\beta^k(Z)$ move in opposite directions on α and lie in different copies of $\alpha_{\mathcal{F}}$.

It follows from this observation and the definition (3.1) of the equating mapping that, as Z moves along α, the points $\vartheta(Z)$ and $\vartheta(j_\beta^k(Z))$ move in the same direction along $\mathbb{T}$. Now the result of the lemma follows from the j_β-invariance of μ and the normalization condition.

It follows from Propositions 2.1 and 2.2 that for intersecting conics we have $c_\beta = \pm\mu(Aa)$ (the points A and a are as in Proposition 2.1), and in the case when one conic lies inside the other $c_\beta = \pm\mu(AB)$ (the points A and B are as in Proposition 2.2).

Proof of the Great Poncelet Theorem. By the definition of an inscribed/circumscribed polygon the number of tangencies of its sides with the conics $\beta_1, \dots, \beta_n$ which are exterior with respect to α is even. Therefore, applying to a vertex X of the polygon the Poncelet maps $j_{\beta_i}^{\varepsilon_i}$ which take it each time to the vertex next in order, at the nth step we arrive at a point in $\widetilde{\alpha}_{\mathcal{F}}$ which coincides with the starting vertex on one of the copies of α_β (because we only go over from one copy to another in the case of an exterior tangency). Now we apply the mapping ϑ to both sides of the equality

$$j_{\beta_n}^{\varepsilon_n} \circ \dots \circ j_{\beta_1}^{\varepsilon_1}(X) = X$$

and use property (3.2). Then we obtain

$$\rho_{d_n} \circ \dots \circ \rho_{d_1}(x) = x, \quad \text{where} \ \ x = \mu(E, X), \quad d_i = (-1)^{\varepsilon_i} c_{\beta_i},$$

so that

$$d_n \oplus \dots \oplus d_1 = 0, \tag{3.3}$$

$$\rho_{d_n} \circ \dots \circ \rho_{d_1} \equiv id_{\mathbb{T}}. \tag{3.4}$$

The Great Poncelet Theorem is equivalent to the identity

$$\xi := j_{\beta_n}^{\varepsilon_n} \circ \dots \circ j_{\beta_1}^{\varepsilon_1} \equiv id_{\widetilde{\alpha}_{\mathcal{F}}}.$$

When the conics in $\mathcal{F}$ intersect at two points with odd multiplicity, the mapping ϑ is bijective and the identity $\xi \equiv id_{\widetilde{\alpha}_\mathcal{F}}$ is an immediate consequence of (3.4). In the general case it follows from (3.4) that for any $Y \in \alpha_\mathcal{F}$ we have one of the following two statements:

1) $\xi(Y) = Y$ and the points Y and $\xi(Y)$ belong to the same copy of $\alpha_\mathcal{F}$;

2) $\mu(Y,E) = \mu(E,\xi(Y))$ and the points Y and $\xi(Y)$ belong to different copies of $\alpha_\mathcal{F}$.

We shall show that only 1) can occur in both remaining cases of mutual position of the conic.

If the conics in $\mathcal{F}$ intersect at four points, then $\alpha_\mathcal{F}$ consists of two arcs and the Poncelet map j_{β_i} can take you from one arc to the other only when β_i is a hyperbola (perhaps degenerate into a pair of intersecting lines). Since an inscribed/circumscribed polygon exists, the number of hyperbolae in the set $\beta_1, \dots, \beta_n$ must be even, the points Y and $\xi(Y)$ lie on the same arc of $\alpha_\mathcal{F}$, and the equality $\mu(Y,E) = \mu(E,\xi(Y))$ is impossible.

If the conics in $\mathcal{F}$ are disjoint, then either each β_i lies inside α or each β_i lies outside α. Then the relevant tangencies are always interior or always exterior. Since an inscribed/circumscribed polygon exists, the number of exterior tangencies must be even, and the points Y and $\xi(Y)$ lie in the same of the two copies $\alpha^1_\mathcal{F}$ and $\alpha^2_\mathcal{F}$ of the set $\alpha_\mathcal{F}$, which is only possible when 1) occurs.

We see that if some inscribed/circumscribed polygon exists, then it can have a vertex at any point in α such that a tangent to β_1 can be drawn from this point. Moreover, as we can rearrange the terms in (3.3) or multiply this equality by -1, the sides of the polygon can be tangent to $\beta_1, \dots, \beta_n$ in an arbitrary order corresponding to an arbitrary permutation σ of the set $\{1, \dots, n\}$, and either of the two points of intersection with α of an arbitrary tangent to β_{σ_1} can be the first vertex of the polygon.

§ 4. The case of a degenerate conic

In this section we treat the case when α is a degenerate conic equal to a pair of straight lines and its vertex lies outside the conics in the pencil $\mathcal{F}$. Then the function

$$f(X) = \frac{1}{|\nabla \mathcal{P}_\alpha(X)| \sqrt{\mathcal{P}_\beta(X)}}$$

has a nonintegrable singularity at the vertex P of α and $\mu(\alpha) = \infty$, so we cannot normalize the measure and construct an equating mapping in a similar way to in § 3.

Let l_1 and l_2 be the lines forming α. As before, we look at the copies $\alpha^1_\mathcal{F}$ and $\alpha^2_\mathcal{F}$ of the set $\alpha_\mathcal{F}$ and denote their union by $\widetilde{\alpha}_\mathcal{F}$. We take two points

$$E_1 \in l_1 \cap \alpha_\mathcal{F}, \qquad E_2 \in l_2 \cap \alpha_\mathcal{F},$$

neither of which coincides with P. Each point Z in α which is distinct from P lies on one of the lines l_1 and l_2, for instance, on l_1. The points Z and E_1 partition l_1 into two 'projective intervals', one being an ordinary closed interval and the other being its complement. The measure μ is finite on the interval not containing P; we denote this interval by $[E_1, Z]$. Fixing some orientation of l_1 and l_2, the interval $[E_1, Z]$ will be positively oriented if the direction from E_1 to Z orients l_1, and it is negatively oriented otherwise. We set $\tau(Z) = \pm 1$ depending on the orientation of $[E_1, Z]$ ($+1$ for the positive orientation and -1 for the negative one).

Taking a pair of parallel lines o_1 and o_2 we select on them the same orientation, defined by some directing vector $\overrightarrow{e}$. We map the points E_1 and E_2 to some points $X_1 \in o_1$ and $X_2 \in o_2$ such that $X_1X_2 \perp \overrightarrow{e}$. Now we produce a mapping ϑ which takes $\widetilde{\alpha}_{\mathfrak{F}}$ to the pair of lines o_1, o_2.

Assume that a point Z in l_k lies in the copy $\alpha^i_{\mathfrak{F}}$, $i, k = 1, 2$. Then we define $\vartheta(Z)$ to be the point in o_k such that

$$\overrightarrow{X_k\vartheta(Z)} = (-1)^i \tau(Z)\mu[E_k, Z]\overrightarrow{e}, \tag{4.1}$$

where we set $\mu(\alpha \setminus \alpha_\beta) = 0$.

We see that Z as a point in $\alpha^1_{\mathfrak{F}}$ and Z as a point in $\alpha^2_{\mathfrak{F}}$ are taken to points on o_1 (or o_2) symmetric with respect to X_1 (or X_2).

Fixing an orientation of a straight line determines the direction of the normal to it. We mark points in the half-plane (bounded by the straight line under consideration) containing the normal vector erected from this line by the 'plus' sign and we mark points in the other half-plane by the 'minus' sign. Two oriented lines form two vertical angles. We say that the angle marked with different signs is *negative*.

Remark 4.1. We defined above (see the introduction) the exterior of a degenerate conic equal to a pair of straight lies as the interior of either of the corresponding vertical angles. We shall now assume that we have fixed the orientation of l_1 and l_2 so that the exterior of α coincides with the negative angle.

Now we pick E_1 and E_2 as follows. If the conics in the pencil do not intersect l_i, then $E_i := l_i \cap p$, where p is the polar of P with respect to any conic in $\mathfrak{F}$ (the polar of P with respect to any conic in the pencil is the same). If these conics intersect l_i, then we take either of the points of intersection for E_i. Then we have the following result.

Proposition 4.2. *Each conic β in $\mathfrak{F}$ is associated with a number $c_\beta := \frac{1}{2}\mu[G, K]$ such that the equating mapping transforms each of the Poncelet maps j^1_β and j^2_β into the composite of a translation by the vector $\pm c_\beta$ and the symmetry S interchanging o_1 and o_2. More precisely, for each $k \in \{1, 2\}$ the following diagrams are commutative*:

$$\begin{array}{ccc} l_1 & \xrightarrow{\;j^k_\beta\;} & l_2 \\ \downarrow \vartheta & & \downarrow \vartheta \\ o_1 & \xrightarrow{\;S\circ\rho_{(-1)^k c_\beta}\;} & o_2, \end{array} \qquad \begin{array}{ccc} l_2 & \xrightarrow{\;j^k_\beta\;} & l_1 \\ \downarrow \vartheta & & \downarrow \vartheta \\ o_2 & \xrightarrow{\;S\circ\rho_{(-1)^{k+1} c_\beta}\;} & o_1 \end{array}$$

Proof. Since μ is invariant under the Poncelet maps, it follows from the definition (4.1) of the equating mapping that to see that the equating transforms the Poncelet maps into translations we only need to verify that, as a point Z moves along α, the points $\vartheta(Z)$ and $\vartheta(j^k_\beta(Z))$ move along the lines o_1 and o_2 in the same direction. The fact that these motions are the translations by $\pm c_\beta$ follows from Proposition 2.5.

It follows from Remark 4.1 that if the straight line $Zj^k_\beta(Z)$ makes an interior tangency with β, then as Z moves on α in the positive direction, the point $j^k_\beta(Z)$

also moves along α in the positive direction. Now since Z and $j^k_\beta(Z)$ lie on the same copy of α, the points $\vartheta(Z)$ and $\vartheta(j^k_\beta(Z))$ move along o_1 and o_2 in the same direction. But if $Zj^k_\beta(Z)$ makes an exterior tangency with β, then as Z moves on α in the positive direction, the point $j^k_\beta(Z)$ moves along α in the negative direction. However, Z and $j^k_\beta(Z)$ lie in distinct copies of α, so the points $\vartheta(Z)$ and $\vartheta(j^k_\beta(Z))$ move along the lines o_1 and o_2 in the same direction.

Now the Great Poncelet Theorem in the case of a degenerate quadric α follows from Proposition 4.2.

Corollary 4.3. *If Poncelet trajectories of conics α and β, where α is degenerate and equal to a pair of straight lines, are closed, then the vertex of α lies inside β. Otherwise (for example, when α and β are disjoint) Poncelet trajectories converge to the vertex of α.*

Proof. Let $Z_1Z_2\dots Z_n\dots$ be a Poncelet trajectory of α and β. For definiteness assume that the point Z_2 is obtained from Z_1 by applying the map j^1_β; then $Z_3 = j^2_\beta(Z_2)$, $Z_4 = j^1_\beta(Z_3)$, $\dots$, that is, in this sequence the Poncelet maps j^1_β and j^2_β are applied in alternation.

Now we look at the image $Z'_1Z'_2\dots Z'_n\dots$ of the Poncelet trajectory $Z_1Z_2\dots Z_n\dots$ under the equating mapping ϑ. The points Z'_i with even indices lie on one of the lines o_1 and o_2, and the points with odd indices lie on the other line. By Proposition 4.2 the point Z'_{i+1} is obtained from Z'_i by translating by a certain vector, the same for all i, and then using the symmetry S, so that the sequence Z'_i converges to the point at infinity of the lines l_1 and l_2. This is equivalent to the convergence of the sequence Z_i to P.

§ 5. Description of all *j*-invariant measures

In the proof of the Great Poncelet Theorem we used a particular measure (defined in Theorem 1.5) which is invariant under the Poncelet map and turns out also to be $\mathcal{F}$-universal. It is natural to ask if there exist other invariant measures and universal measures.

The next theorem gives a complete classification of Borel measures on conics which are invariant under the Poncelet maps, and of universal measures.

Theorem 5.1. *Let α and β be nondegenerate conics.*

1. *If Poncelet trajectories of α and β are not closed, then there exists a unique j_β-invariant Borel measure. Furthermore, this measure is absolutely continuous with respect to the Lebesgue measure and $\mathcal{F}$-universal.*

2. *If Poncelet trajectories of α and β are closed, then there exist infinitely many invariant Borel measures. Every such measure μ is uniquely determined by its definition on an arbitrary arc of the form (a,b), where b is a point in the j_β-orbit of a such that the arc (a,b) contains no other points in this orbit.*

3. *An $\mathcal{F}$-universal measure on α is uniquely defined; it coincides with the measure defined in Theorem 1.5.*

Proof. In fact, each j_β-invariant Borel measure ν generates the Borel measure $\lambda = \nu\circ\vartheta^{-1}$ on $\mathbb{T}$ which is invariant under the translation ρ_{c_β}.

If Poncelet trajectories are not closed, then $j_\beta^n \neq id_\alpha$ for each $n \in \mathbb{N}$. Applying ϑ to this inequality in accordance with (3.2) we obtain $\rho_{c_\beta}^n \neq id_{\mathbb{T}}$, which is equivalent to the inequality $nc_\beta \neq 0_{\mathbb{T}}$ or $nc_\beta \notin \mathbb{Z}$, so that c_β is irrational.

Each Borel measure λ on $\mathbb{T}$ is a distribution:

$$(\lambda, \varphi) = \int \varphi\, d\lambda, \qquad \varphi \in D,$$

where D is the space of test functions. Each distribution $\lambda \in D'$ can be represented uniquely by a Fourier series converging in D':

$$\lambda(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x}.$$

If the measure λ is invariant under the translation by a vector c, then

$$\sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x} = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k (x+c)} = \sum_{k \in \mathbb{Z}} (a_k e^{2\pi i k c}) e^{2\pi i k x}.$$

Since a function can be represented uniquely by a Fourier series, for each k we either have $a_k = 0$ or $e^{2\pi i k c} = 1$. If c is irrational, then the last equality holds only for $k = 0$. Thus $\lambda(x) = a_0$, that is, the measure is unique up to normalization.

In the case when trajectories are closed, c_β is rational; let it be equal to an irreducible fraction m/n, where n is the number of segments in the trajectory, that is, the number of points in the j_β-orbit of $a \in \widetilde{\alpha}_{\mathcal{F}}$. Then we have the decomposition

$$\widetilde{\alpha}_{\mathcal{F}} = \bigsqcup_{k=0}^{n-1} j_\beta^k([a, b)),$$

where j_β^k is the kth power of j_β in the sense of composite maps. Now an arbitrary measure μ on $[a, b)$ can be extended uniquely to the whole of α as an invariant measure.

We have already shown that an $\mathcal{F}$-universal measure exists. Its uniqueness is a consequence of part 1. of the theorem. In fact, let X and Y be points in α such that $\mu(X, Y)$ is irrational. The pencil $\mathcal{F}$ contains a conic γ tangent to the line XY. Then Poncelet trajectories of γ are not closed. Each $\mathcal{F}$-universal measure is j_γ-invariant, and it follows from part 1. that the j_γ-invariant measure is unique.

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