Liouville’s theorem for pedants

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Abstract

We discuss Liouville’s theorem on the evolution of an ensemble of classical particles, using language friendly to differential geometers.

1 Liouville’s theorem, classical statement

The classical statement of Liouville’s theorem [4] is that an ‘ensemble’ of particles described by a density function \( \rho = \rho(q_i, p_i, t) \) evolves in time according to the equation:

\[
\frac{\partial \rho}{\partial t} + \sum_i \left( \frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = 0,
\]

where \( q_i, p_i \) are generalised coordinates on phase space and \( t \) is time. This note discusses its statement in modern language and tries to clarify the question to which Liouville’s theorem is the answer.

2 Classical mechanics

Recall that the modern model of classical mechanics is a symplectic manifold \((M, \omega)\) together with a function:

\[ H : M \to \mathbb{R}, \]

the Hamiltonian.

This data determine a vector field \( X_H \) on \( M \), the dual of \( dH \) under identification of \( TM \) and \( T^*M \) using \( \omega \). The defining equation is thus:

\[ dH = i_{X_H} \omega, \]
where \( i_{X_H} \omega = \omega(X_H, \cdot) \) is the interior product. \( X_H \) is known as the Hamiltonian vector field associated to \( H \).

We obtain time evolution for the system \((M, \omega, H)\) by integrating \( X_H \) to its flow. I.e., the 1-parameter group of diffeomorphisms\(^1\):

\[
\phi : M \times \mathbb{R} \to M,
\]

which generates \( X_H \), constitutes time evolution. In other words, the integral curves of \( X_H \) represent physical motion according to \( H \).

3 Liouville’s theorem, modern statements

Consider the following:

**Proposition 3.1.** Let \((M, \omega)\) be a symplectic manifold and \( X \) a vector field on \( M \), then the following are equivalent:

- \( X \) is locally Hamiltonian.
- \( \mathcal{L}_X \omega = 0 \).
- The flow \( \phi_t = \phi(\cdot, t) \) associated to \( X \) consists of symplectomorphisms.

where \( \mathcal{L} \) is the Lie derivative.

**Proof.** This follows easily from two characteristic properties of the Lie derivative, namely \( \mathcal{L}_X = di_X + i_X d \) (together with the Poincaré lemma) as well as

\[
\mathcal{L}_X = \lim_{h \to 0} \left( \frac{\phi_t^* - id}{h} \right) \quad \text{(use} \quad \frac{d}{dt} \phi_t^* = \phi_t^* \mathcal{L}_X \quad \text{to deduce} \quad \phi_t^* \omega \quad \text{is constant).} \]

For some people, a modern statement of Liouville’s theorem is:

**Corollary 3.2.** If \((M, \omega, H)\) is a physical system then \( \mathcal{L}_{X_H} \omega = 0 \).

Although the result is definitely relevant, it is somewhat distant from the classical statement: there is no sight of anything playing the role of \( \rho \).

Others view a modern version of Liouville’s theorem to be:

**Corollary 3.3.** If \((M, \omega, H)\) is a physical system then \( \phi_t \) consists of symplectomorphisms.

A popular view [1, 3] is to regard the following slightly weaker result as the content of Liouville’s theorem:

\(^1\)Technically \( \phi \) may only be locally defined, i.e., it is a sheaf. We will not emphasise this since we would gain nothing by it here.
Corollary 3.4. If \((M, \omega, H)\) is a 2\(n\)-dimensional physical system then \(\phi_t\) preserves \(\omega^n\), i.e., time evolution preserves volume in phase space.

It’s mostly just a matter of taste but my preference is to reserve Liouville’s name for a proposition that actually mentions \(\rho\).

Consider then a physical system about whose initial state we have incomplete information. Instead of our usual model of initial data as a distinguished point of phase space, we generalise and model initial data as a probability measure \(\rho_0\) on phase space for some function:

\[ \rho_0 : M^{2n} \to \mathbb{R}, \]

which represents our information (and satisfies \(\int_M \rho_0 \omega^n = 1\)).

Consider time evolution for \(\rho_0\). Because classical mechanics is perfectly deterministic, the probability density \(\rho(x, t)\) of a state \(x \in M\) at any time \(t\) is uniquely determined: just follow the curve representing physical motion through \(x\) back for \(t\) units of time, reaching a point \(x_0\), say. We must have:

\[ \rho(x, t) = \rho_0(x_0). \]

In other words prescribing the likelihood of an initial state is the same as prescribing the likelihood of the full history of physical motion through that state. Let’s capture this in a definition:

**Definition 3.5.** Let \((M, \omega, H)\) be a physical system and let \(\rho : M \times \mathbb{R} \to \mathbb{R}\). We say \(\rho\) obeys Newton’s laws if \(t \mapsto \rho(\alpha(t), t)\) is constant for all integral curves \(\alpha\) of \(X_H\).

The question then is: which functions \(\rho\) obey Newton’s laws? The answer is Liouville’s theorem:

**Proposition 3.6.** Let \((M, \omega, H)\) be a physical system and let \(\rho : M \times \mathbb{R} \to \mathbb{R}\). Then \(\rho\) obeys Newton’s laws iff

\[ \frac{\partial \rho}{\partial t} + \{\rho, H\} = 0, \]

where \(\{\cdot, \cdot\}\) is the Poisson bracket of \((M, \omega)\).

**Proof.** Differentiate \(t \mapsto \rho(\alpha(t), t)\), use Hamilton’s equations and require that the result be 0. \(\square\)

Note that a tautological restatement of the condition on \(\rho\) in definition 3.5 is simply that:

\[ \phi_t^* \rho_t = \rho_0 \quad \text{for all} \ t, \]

where \(\rho_t = \rho(\cdot, t)\) and as usual \(\phi_t\) is the flow generating \(X_H\). By corollary 3.4 we can thus restate proposition 3.6 as:
Proposition 3.7. Let \((M^{2n}, \omega, H)\) be a physical system with flow \(\phi_t\) and let \(\rho : M \times \mathbb{R} \to \mathbb{R}\). Then \(\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0\) iff
\[
\phi^*_t \mu_t = \mu_0 \quad \text{for all } t,
\]
where \(\mu_t = \rho(\cdot, t) \omega^n\) is the probability measure on \(M\) at time \(t\).

We can thus regard Liouville’s theorem as the statement that classical mechanics remains ergodic, even when considering ensembles of particles. I suspect this is the reason for the popularity of regarding corollary 3.4 as a modern statement of Liouville’s theorem even though they’re not quite the same result.

There is one final point, due to Gibbs [2] worth mentioning: we can regard Liouville’s differential equation (1) as a physical continuity equation for probability density flowing through phase space like a fluid with velocity \(X_H\), and without sinks or sources.

The classical continuity equation for a fluid with density \(\rho\) and velocity vector field \(v\) expresses the local conservation of mass and is usually written:
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.
\]

This can be seen to be analogous to (1) because a symplectic manifold carries a natural differential operator analogous to the Riemannian divergence operator appearing in (2).

Indeed a symplectic manifold carries a natural symplectic star operator:
\[
\star : \Lambda^k \simeq \Lambda^{2n-k},
\]
defined in exactly the same way as the better-known Hodge star operator from metric geometry. We also have the formal adjoint (wrt \(\omega\)) of the exterior derivative:
\[
d^* = (-1)^k \star d\star : \Omega^{k+1} \to \Omega^k.
\]
Many of the properties familiar from metric geometry still hold (e.g., \(d^* d^* = 0\)) but a little linear algebra reveals an important difference between the symplectic and Hodge stars: the would-be symplectic ‘Laplacian’ vanishes\(^2\), i.e., \(d\) and \(d^*\) anti-commute. In particular \(d^* df = 0\) for any function \(f\). It follows that we can express the Poisson bracket as:
\[
\{f, g\} = d^* (f dg),
\]
for any functions \(f, g\). The operator \(d^*\) is symplectic divergence.

We thus have Gibbs’s statement of Liouville’s theorem in modern language:
\(^2\)It is nevertheless still possible to do ‘Hodge theory’ on a symplectic manifold, see [5].
Proposition 3.8. Let $(M, \omega, H)$ be a physical system and let $\rho : M \times \mathbb{R} \to \mathbb{R}$. Then $\rho$ obeys Newton’s laws iff

$$\frac{\partial \rho}{\partial t} + d^*(\rho dH) = 0.$$ 

References


