Liouville's theorem for pedants

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Abstract

We discuss Liouville's theorem on the evolution of an ensemble of classical particles, using language friendly to differential geometers.

1 Liouville's theorem, classical statement

The classical statement of Liouville's theorem [4] is that an 'ensemble' of particles described by a density function $\rho = \rho(q_i, p_i, t)$ evolves in time according to the equation:

$$\frac{\partial \rho}{\partial t} + \sum_i \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = 0,$$

where q_i, p_i are generalised coordinates on phase space and t is time. This note discusses its statement in modern language and tries to clarify the question to which Liouville's theorem is the answer.

2 Classical mechanics

Recall that the modern model of classical mechanics is a symplectic manifold (M, ω) together with a function:

$$H : M \to \mathbb{R},$$

the Hamiltonian.

This data determine a vector field X_H on M, the dual of dH under identification of TM and T^*M using ω. The defining equation is thus:

$$dH = i_{X_H}\omega,$$

where $i_{X_H}\omega = \omega(X_H, \cdot)$ is the interior product. X_H is known as the Hamiltonian vector field associated to H.

We obtain time evolution for the system (M, ω, H) by integrating X_H to its flow. I.e., the 1-parameter group of diffeomorphims[1]:

$$\phi : M \times \mathbb{R} \to M,$$

which generates X_H, constitutes time evolution. In other words, the integral curves of X_H represent physical motion according to H.

3 Liouville's theorem, modern statements

Consider the following:

Proposition 3.1. *Let (M, ω) be a symplectic manifold and X a vector field on M, then the following are equivalent:*

- *X is locally Hamiltonian.*
- $\mathcal{L}_X\omega = 0$.
- *The flow $\phi_t = \phi(\cdot, t)$ associated to X consists of symplectomorphisms.*

where $\mathcal{L}$ is the Lie derivative.

Proof. This follows easily from two characteristic properties of the Lie derivative, namely $\mathcal{L}_X = di_X + i_X d$ (together with the Poincaré lemma) as well as $\mathcal{L}_X = \lim\limits_{h\to 0}\left(\frac{\phi_h^* - id}{h}\right)$ (use $\frac{d}{dt}\phi_t^* = \phi_t^*\mathcal{L}_X$ to deduce $\phi_t^*\omega$ is constant). □

For some people, a modern statement of Liouville's theorem is:

Corollary 3.2. *If (M, ω, H) is a physical system then $\mathcal{L}_{X_H}\omega = 0$.*

Although the result is definitely relevant, it is somewhat distant from the classical statement: there is no sight of anything playing the role of ρ.

Others view a modern version of Liouville's theorem to be:

Corollary 3.3. *If (M, ω, H) is a physical system then ϕ_t consists of symplectomorphisms.*

A popular view [1, 3] is to regard the following slightly weaker result as the content of Liouville's theorem:

[1] Technically ϕ may only be locally defined, i.e., it is a sheaf. We will not emphasise this since we would gain nothing by it here.

Corollary 3.4. *If (M, ω, H) is a $2n$-dimensional physical system then ϕ_t preserves ω^n, i.e., time evolution preserves volume in phase space.*

It's mostly just a matter of taste but my preference is to reserve Liouville's name for a proposition that actually mentions ρ.

Consider then a physical system about whose initial state we have incomplete information. Instead of our usual model of initial data as a distinguished point of phase space, we generalise and model initial data as a probability measure $\rho_0\omega^n$ on phase space for some function:

$$\rho_0 : M^{2n} \to \mathbb{R},$$

which represents our information (and satisfies $\int_M \rho_0\omega^n = 1$).

Consider time evolution for ρ_0. Because classical mechanics is perfectly deterministic, the probability density $\rho(x, t)$ of a state $x \in M$ at any time t is uniquely determined: just follow the curve representing physical motion through x back for t units of time, reaching a point x_0, say. We must have:

$$\rho(x, t) = \rho_0(x_0).$$

In other words *prescribing the likelihood of an initial state is the same as prescribing the likelihood of the full history of physical motion through that state.* Let's capture this in a definition:

Definition 3.5. *Let (M, ω, H) be a physical system and let $\rho : M \times \mathbb{R} \to \mathbb{R}$. We say ρ obeys Newton's laws if $t \mapsto \rho(\alpha(t), t)$ is constant for all integral curves α of X_H.*

The question then is: which functions ρ obey Newton's laws? The answer is Liouville's theorem:

Proposition 3.6. *Let (M, ω, H) be a physical system and let $\rho : M \times \mathbb{R} \to \mathbb{R}$. Then ρ obeys Newton's laws iff*

$$\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0, \tag{1}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket of (M, ω).

Proof. Differentiate $t \mapsto \rho(\alpha(t), t)$, use Hamilton's equations and require that the result be 0. □

Note that a tautological restatement of the condition on ρ in definition 3.5 is simply that:

$$\phi_t^* \rho_t = \rho_0 \qquad \text{for all } t,$$

where $\rho_t = \rho(\cdot, t)$ and as usual ϕ_t is the flow generating X_H. By corollary 3.4 we can thus restate proposition 3.6 as:

Proposition 3.7. *Let (M^{2n}, ω, H) be a physical system with flow ϕ_t and let $\rho : M \times \mathbb{R} \to \mathbb{R}$. Then $\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0$ iff*

$$\phi_t^* \mu_t = \mu_0 \qquad \textit{for all } t,$$

where $\mu_t = \rho(\cdot, t)\omega^n$ is the probability measure on M at time t.

We can thus regard Liouville's theorem as the statement that classical mechanics remains ergodic, even when considering ensembles of particles. I suspect this is the reason for the popularity of regarding corollary 3.4 as a modern statement of Liouville's theorem even though they're not quite the same result.

There is one final point, due to Gibbs [2] worth mentioning: we can regard Liouville's differential equation (1) as a physical *continuity equation* for probability density flowing through phase space like a fluid with velocity X_H, and without sinks or sources.

The classical continuity equation for a fluid with density ρ and velocity vector field v expresses the local conservation of mass and is usually written:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0. \tag{2}$$

This can be seen to be analogous to (1) because a symplectic manifold carries a natural differential operator analogous to the Riemannian divergence operator appearing in (2).

Indeed a symplectic manifold carries a natural symplectic star operator:

$$\star : \wedge^k \simeq \wedge^{2n-k},$$

defined in exactly the same way as the better-known Hodge star operator from metric geometry. We also have the formal adjoint (wrt ω) of the exterior derivative:

$$d^* = (-1)^k \star d \star : \Omega^{k+1} \to \Omega^k.$$

Many of the properties familiar from metric geometry still hold (e.g., $d^{*2} = 0$) but a little linear algebra reveals an important difference between the symplectic and Hodge stars: the would-be symplectic 'Laplacian' vanishes[2], i.e., d and d^* anti-commute. In particular $d^* df = 0$ for any function f. It follows that we can express the Poisson bracket as:

$$\{f, g\} = d^*(f dg),$$

for any functions f, g. The operator d^* is symplectic divergence.

We thus have Gibbs's statement of Liouville's theorem in modern language:

[2]It is nevertheless still possible to do 'Hodge theory' on a symplectic manifold, see [5].

Proposition 3.8. *Let (M, ω, H) be a physical system and let $\rho : M \times \mathbb{R} \to \mathbb{R}$. Then ρ obeys Newton's laws iff*

$$\frac{\partial \rho}{\partial t} + d^*(\rho dH) = 0.$$

References

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