Mathematical loose ends after a year off

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Contents

Abstract

I am about to return to work 'in industry' and so do not expect to have time for mathematics for a long time. Since I have some ideas that I would have worked on if I had had time, I am collecting a few scraps here. I am keeping the focus narrow since I have collected my thoughts on other problems (mostly in gauge theory) elsewhere.

Ideas are listed in increasing order of how speculative they are; towards the end I will allow myself to indulge in blatant speculation.

I am writing these pages mostly for my future self but I will make them publicly available since there is a chance, albeit small, that they may be of use to others. In some cases I have ideas about how to proceed and in others I simply liked the look of a particular problem.

1 K-theory and Landsberg's Clifford module

I believe that my results in [18] suggest certain problems in algebraic geometry to which topological K-theory may be usefully applied. Aside from the

Hartshorne complete intersection conjecture[1] [9], I believe it may be possible to make progress understanding properties of secant-defective varieties using K-theoretic methods.

Amongst the questions in my mind regarding secant-defective varieties is the question of the existence of a smooth variety with secant deficiency $\delta > 8$. This question was first raised in [17]. All we currently know is that any such varieties would not be homogeneous [14]. I have not thought seriously about this.

One thing I did spend a few weeks thinking about was how to connect my results in [18] with the Clifford module structure that Landsberg introduced in [15]. I can more or less see how this should work but I got distracted with other things so my thoughts are incomplete. I summarised my thoughts on this in a letter I sent to Prof. Francesco Russo in January which I have included as an appendix to this document.

One of the issues I mentioned in that letter is the issue of the ABS construction of [3] for bundles of complex quadratic forms. I have not thought seriously about this since last November but I am pretty sure the solution must be simple so I asked the question on MathOverflow:

`http://mathoverflow.net/questions/162073/`

Unfortunately it is a bit technical and, even after offering a bounty, I have not received any answers.

2 Homogeneity conjecture for QELMs

Ionescu and Russo made the following conjecture:

Conjecture 1 ([11], page 959)**.** *Let $X \subset \mathbb{P}^N$ be a secant-defective QELM that is not a hyperplane section and such that* $\mathrm{Pic}(X) \simeq \mathbb{Z}$*, then X is homogeneous.*

As Ionescu and Russo note, this would essentially complete the classification of secant-defective QELMs.

I believe this conjecture should be within range. One avenue that seems worth exploring is whether the methods of Chaput [4, 5] can be applied. Without using Zak's classification, Chaput shows that Scorza varieties (which are all QELMs by Zak [21] chapter VI, corollary 1.5) are homogeneous. He does this by showing that if $X \subset \mathbb{P}^N$ is a k-Scorza variety then a general point

[1]Though the path is not clear, I would at least hope to be able to find an alternative proof of Ionescu and Russo's impressive result in [12].

outside $\mathrm{Sec}^{k-1}(X)$ determines an isomorphism $\mathbb{P}^N \simeq \mathbb{P}^{N*}$ (see [4] proposition 2.1 and [5] proposition 2.2).

Even if the techniques of Chaput cannot be used, I still think the conjecture is worth some time.

3 Morse-theoretic approach to Severi variety dimensions

In [6], Eels and Kuiper proved that any compact, connected smooth manifold supporting a Morse function with exactly three critical points necessarily has dimension 2, 4, 8 or 16. (Indeed, they proved an even stronger result, see e.g., [6] corollary, page 22.) Furthermore amongst the tools they used were results of Adams closely related to the Hopf-invariant-one problem. Also, since Atiyah and Adam's paper [1], we know that the Hopf-invariant-one problem is best approached via K-theory.

In view of the connections between projective planes, Severi varieties, division algebras and K-theory I wondered if it might be possible to take a Morse-theoretic approach to the proof that a Severi variety must have dimension 2, 4, 8 or 16. The idea would be to show that the ambient geometry of a Severi variety allows one to show that it supports a Morse function of a special type which, in the spirit of Eels-Kuiper, one hopes would restrict the dimension. Of course we're talking about Severi varieties over the complex numbers (which have real dimensions 4, 8, 16, 32) so I guess we'd look for a 'complex Morse function'. This would put us in the domain of Picard-Lefschetz theory (about which I know almost nothing).

Last Summer, I spent a couple of days trying to see how one might use the Severi geometry to construct a holomorphic function from the Severi variety to $\mathbb{P}^1$ with this sort of idea in mind. I tended to think a good starting point was to fix a line outside the secant variety (meeting it transversely) and to identify the three points where it meets the secant variety with $0, 1, \infty$ (in some order) so that the line becomes $\mathbb{P}^1$. One criterion that I thought might be useful would be if the sought-for construction also worked over the reals so that the function would then be a real Morse function with three critical points. I tried lots of ideas but couldn't find what I wanted.

One problem is that I know almost nothing of Picard-Lefschetz theory. Indeed, unlike the real case, I don't know what happens if you have complex Morse functions with just two or three critical points. I wonder what is known here.

I discovered the much simpler approach to the Severi variety dimension restriction in [18] before I finished thinking about this so I didn't really look back but I would still like to know if a Morse-theoretic / Picard-Lefschetz -type approach could work.

4 Kervaire invariant problem

The Kervaire invariant is a homotopy invariant associated to a $(4k + 2)$-dimensional framed smooth manifold, and takes values in $\{0, 1\}$. It can be defined as an algebraic invariant of the middle-dimensional cup product skew form. See for example Snaith's book [20] for background (as well as why the invariant matters).

By a result of Browder, the only dimensions in which the invariant is possibly non-zero are of the form $2^n - 2$. Thanks to a dramatic breakthrough of Hill, Hopkins and Ravenel (see e.g., [10]) the set of dimensions in which it is possibly non-zero was further narrowed down to $2, 6, 14, 30, 62, 126$. Furthermore it is known that there are framed manifolds of dimensions $2, 6, 14, 30, 62$ with Kervaire invariant one. Thus the only unknown dimension is 126.

Michael Atiyah has expressed the hope that there is a link between the Kervaire invariant problem and the geometries associated to the Freudenthal magic square (see [16] for a good account of these geometries as well as Freudenthal's paper [7]). The dimensions at least seem to match up provided that there is indeed a 126-dimensional (framed) manifold with Kervaire invariant one. However forging the link must be difficult since Atiyah mentioned in an email that he had spent a little time working on the problem but had not cracked it!

Obviously, working on something that Atiyah did not crack is not a recipe for success and so I did not think much about this problem. Nevertheless, I will outline some extremely speculative thoughts that I might have pursued if I were braver.

The geometries associated to the second row of the magic square are the four Severi varieties. The incidence varieties of points on a 'line' (or 'lines' through a point — there is a duality) are the entry loci of the Severi varieties. Since I discovered that a hyperplane section of these entry loci carry unusual topological data (namely a special relation in their K-theory) I wondered if taking a hyperplane section of an incidence variety of a geometry associated to a location in the magic square, might be the right flavour of construction to get manifold with Kervaire invariant one[2].

Now the hyperplane sections of the incidence varieties of Severi varieties are non-singular quadrics of complex dimensions $0, 1, 3, 7$ and the low-

[2] I did warn this was blatant speculation!

dimensional examples of manifolds of Kervaire invariant one associated to division algebras are $S^k \times S^k$, for $k = 1, 3, 7$. Even if we could only say something about the second row of the magic square, I still think it would be significant if we could construct $S^k \times S^k$ (with appropriate framing) somehow from the magic square. Since this is like a split-signature phenomenon, I wondered what would happen if we looked at the incidence varieties of the geometries associated to the second row of a split-signature magic square. This would be the geometries associated to $A' \otimes B$ or $A' \times B'$ or $A \otimes B'$ where A, B are one of the four division algebras and A', B' are their split versions.

The first thing to do would be to construct the geometries[3] in the split cases like Landsberg and Manivel [16] do in the non-split case and then the examine the topology of the incidence varieties, trying to find the $S^k \times S^k$.

A Letter to Prof. Russo

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Dear Francesco,

I wanted to share a few thoughts with you regarding secant-defective varieties. I spent several weeks last October and November thinking about the ideas I sketch below but since then my time has been taken up by other (largely non-mathematical) projects.

As I will almost-certainly soon return to a career in finance, it is unlikely I will be able to finish these investigations in the foreseeable future. However I am confident some of these ideas can be made to work (and hopefully to

[3]Constructing the Lie algebras is easy enough.

obtain new results) so I thought it would be worth spending a few hours writing this letter.

Needless to say I am sure you already have many ongoing research projects and so you may not have time to think seriously about this material, but I hoped you might know somebody who could be interested. Please feel free to share this correspondence with anybody you think might care to take a look; I would be happy to correspond with any such person and to expand on the below.

As to the ideas I have in mind, my motivation was a desire to connect the ideas in Landsberg's paper [15] with the techniques used in my paper [18]. In [15] Landsberg shows how to obtain the dimension restriction for Severi varieties by using the second fundamental form to construct a certain Clifford module. However ever since Atiyah, Bott and Shapiro's seminal paper [3] it has been known that there are deep connections between Clifford modules and K-theory. I believe it is worth exploring these connections in the context of secant-defective varieties. In the worst case, one can show how Landsberg's Clifford module structure provides a third proof of the divisibility property for LQEL manifolds (admittedly we probably have enough proofs of this). In the best case it may be possible that the Atiyah-Bott-Shapiro ideas make possible the use of powerful K-theoretic methods for obtaining new results for general secant-defective varieties via Landsberg's Clifford module structure.

Restricting to Severi varieties, what Landsberg essentially[4] shows [15, theorem 6.26] is the following:

Proposition 2. *Let $X \subset \mathbb{P}^N$ be a Severi variety, $Q \subset X$ be a general entry locus, $F \subset Q$ a general tangent locus and $x \in F$ a general point. Then T_xF carries a natural non-degenerate quadratic form and the fibre of the normal bundle $N^x_{Q|X}$ is a Clifford module for the Clifford algebra $Cl(T_xF)$.*

It is not immediately clear when the hypotheses of Landsberg's result hold (though certainly, as stated, for Severi varieties) and which are necessary. Also, he restricts to the case when the secant variety has codimension 1 and I believe this restriction can at least be lifted in the presence of LQEL geometry. Assuming I do not err, proposition 2 in fact holds for any sufficiently-defective LQEL manifold. The divisibility property then follows in view of the following:

Proposition 3. *Suppose we have a complex vector space of dimension n carrying a non-degenerate quadratic form and suppose that we also have an*

[4] Unfortunately, although [15] is full of wonderful ideas, it also suffers from an unusually-large number of typos, including the statement of the key Clifford-module structure. Many of these are corrected in the book [13].

m-dimensional module for the associated Clifford algebra. Then:

$$2^{[\frac{n}{2}]} \mid m$$

This proposition (which Landsberg seems to have avoided using by means of low-dimensional case analysis) is a trivial consequence of the classification of Clifford modules together with basic facts about representations of matrix algebras. (Interestingly, the dependence on the parity of n thus corresponds to the mod-2 periodicity of Morita equivalence classes of complex Clifford algebras, i.e., it is the same 2 as appears in complex Bott periodicity.) So we have the divisibility property for LQELs since $n = \delta - 1$ and $m = n - \delta$ for the Landsberg Clifford module structure.

Now to make the connection with K-theory, we need to use the ideas of Atiyah, Bott and Shapiro (henceforth ABS). ABS establish the following result which is not quite what we need but is a good starting point:

Proposition 4. *Let $V \to X$ be a real vector bundle such that each fibre carries a positive-definite quadratic form and suppose that $W \to X$ is a bundle of graded Clifford modules over the field k ($k = \mathbb{R}$ or $\mathbb{C}$ are both allowed even though V must be real) for the bundle of Clifford algebras $Cl(V)$. Then W determines a natural element of $\tilde{K}_k(Th(V))$ where $Th(V)$ is the Thom space of V and $\tilde{K}_k$ is reduced real or complex K-theory according to k.*

In fact ABS show that this map depends only on the class of W up to a natural equivalence and so the above proposition provides a map that fits into an exact sequence of Abelian K-groups associated to V but this is not relevant here.

Several remarks are in order:

- Proposition 4 deals in terms of *graded* Clifford modules whereas the Landsberg Clifford module is ungraded. This is not a problem as there is a straightforward correspondence between graded and ungraded modules discussed by ABS[5]. For example if V and W are as in proposition 4 except that W is ungraded then $W \oplus W$ is naturally a bundle of graded Clifford modules for the bundle of Clifford algebras associated to the orthogonal direct sum $V \oplus 1$ (with the positive-definite quadratic form on the factor 1).

- Proposition 4 uses *bundles* of Clifford algebras and modules. If we are to apply something like it using the Landsberg Clifford module

[5]ABS discuss the case when X is a point but there does not seem to be a problem for general X.

structure then in the notation of proposition 2 we must let x vary. I believe this is not a problem, although TF or $N_{Q|X}$ may pick up a twisting by $\mathcal{O}(\pm 1)$. In fact I expect it will be $N_{Q|X} \oplus N_{Q|X}(-1)$ that carries the structure of a bundle of graded Clifford module structures. Also since it is $TF(-1)$ that carries a natural quadratic form and since:

$$TQ(-1) = TF(-1) \oplus \mathcal{O}$$

is a natural orthogonal decomposition on F, I expect it will be the bundle of graded Clifford modules associated to $TQ(-1)$ that will play a role.

- Proposition 4 applies to *real* Clifford algebras whereas the Landsberg Clifford algebra is complex. This changes the setup in a non-trivial way which I will discuss below.

Regarding the last point mentioned above, in concrete terms, the ABS construction (which yields an element of K-theory from a graded Clifford module) does not work when V is a complex vector bundle. Their construction relies on the property that for a positive-definite real quadratic form q we have $q(v) = 0 \iff v = 0$ for a vector v, which of course is false for any quadratic form in the complex case.

Note also that there is a topological obstruction for a complex vector bundle V to carry a non-degenerate quadratic form: it does so iff $V \simeq V^*$. Furthermore when this holds, by choosing a Hermitian metric we get $V \simeq \bar{V}$ and so V is the complexification of a real vector bundle V_R, say, carrying a positive-definite quadratic form. From this point of view, a complex vector bundle carrying a non-degenerate quadratic form is just the complexification of an underlying real vector bundle with non-degenerate quadratic form *except that we have forgotten the signature of the real quadratic form.*

Related to this then is the ABS construction for non-degenerate indefinite-signature real quadratic forms. Again the naïve ABS construction fails to go through since we no longer have the $q(v) = 0 \iff v = 0$ property but in a subsequent paper [2] to ABS [3], Atiyah shows how to make things work. (He shows how real Clifford modules for real quadratic forms of signature (a, b) give elements of a K-group isomorphic to KO^{a-b} of the base.)

In fact the indefinite-signature real case may be enough to deal with the case of the ABS construction for non-degenerate complex quadratic forms since we can decompose such a form into its real and imaginary components q_R, q_I where are split-signature real quadratic forms, if we forget the complex structure. (This seems plausible since q_R and q_I together determine the forgotten complex structure.)

The issue then is that the combined results of [3] and [2] cover the ABS construction for real and complex modules of definite-signature real quadratic forms as well as real modules of indefinite-signature real quadratic forms. However the case of *complex* modules of indefinite-signature real quadratic forms (which as I say, I think may resolve the case of non-degenerate complex quadratic forms) does not seem to have been discussed anywhere.

Notwithstanding the last four paragraphs, I think I may be making a bit of a meal of this. The complex case should be simpler than the real case! I can almost believe that a result along the following lines might be true:

Not-entirely-serious proposition 5. *Let $W \to X$ be a bundle of graded Clifford modules for the bundle of complex Clifford algebras associated to a complex vector bundle with non-degenerate quadratic form. Then $[W] = \operatorname{rank} W$ in $K(X)$ (i.e., [W] is trivial).*

I do not seriously claim this sort of wild speculation is correct (I have not taken time to think properly about it). It would say that the ABS construction for the complex case is essentially trivial but it would be very useful when applied to the Landsberg Clifford module. For example, it would recover the content of remark 2.7 in my paper [18]. Whatever the truth, I am sure that some it must be possible to establish some simple general result which will link up my work with Landsberg's.

Assuming the above can be done the next thing to do is to try and see if it is possible to move beyond LQEL manifolds into general secant-defective manifolds. As I have mentioned, I find the hypotheses of Landsberg's theorem 6.26 rather opaque and I confess not to have scrutinised his arguments to the level of detail that allows me to see their domain of applicability.

At least since Griffiths and Harris [8] it has been known that global algebro-geometric information is encoded in the local differential-geometry that is the second fundamental form (henceforth SFF). Landsberg makes excellent use of this by (amongst other things) focusing his attention on the Kodaira map associated to the SFF at a point, viewed as a linear system. To use your standard notation (as in [19] for example), given a point $x \in X \subset \mathbb{P}^N$ this is the rational map:

$$\tilde{\pi}_x : \mathbb{P}(T_x X) \dashrightarrow W_x \subset \mathbb{P}(N^x_{X|\mathbb{P}^N})$$

By considering the derivative of this map at an appropriate point we thus have an isomorphism from the normal bundle of (an irreducible component of) a fibre of $\tilde{\pi}_x$ and the tangent space of the corresponding point in the image of $\tilde{\pi}_x$. Fixing x but varying the point in the fibre, Landsberg shows that these isomorphisms fit together into a space of isomorphisms parameterised by

T_xF (for a tangent locus F) which he shows satisfy the fundamental Clifford algebra identity. Hence the Clifford module structure.

If Landsberg's construction can be made to work for more general varieties than LQELs (this is far from clear — I have not established how much more than secant-defectiveness he needs to establish the Clifford algebra identity) then, using your notation from [19, definition 2.3.3] I might expect that it is the fibres of the normal bundle of $\Xi = \Xi(X, H)$ (for appropriate H) in X that would carry the interesting algebraic data (e.g., a Clifford module structure?). The dimensions at least look right, especially in view of the refinement of Zak's Linear Normality Theorem that both you [19, equation (3.1.5)] and Landsberg [13, page 135] prove.

Even I find it a bit hard to believe that there could in general be a Clifford module structure on the fibre of the normal bundle $N^x_{\Xi|X}$ for the bundle of Clifford algebras associated to T_xF for a tangent locus F and point $x \in F$ since this would give the extremely-strong-looking divisibility property:

$$2^{[\frac{\delta-1}{2}]} \,\Big|\, n - \xi$$

but as I have said, I believe it must be possible to take these ideas beyond the LQEL setting in some way.

Assuming this could be done, then there would be the task of brining in K-theory, presumably along the lines which I outlined I think will work in the LQEL case. Then finally one hopes there would be a dividend in that we could use K-theory to tackle some open problems.

I apologise for the sketchiness of much of the above, I am trying to tidy up several projects relatively quickly. I admit I have been in two minds as to whether I should even release ideas in such an incomplete form but in the end I decided it could surely do no harm.

Best wishes and happy new year,

Oliver Nash

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