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Re: Clifford modules, K-theory and secant-defective varieties

Dear Francesco,

I wanted to share a few thoughts with you regarding secant-defective varieties. I spent several weeks last October and November thinking about the ideas I sketch below but since then my time has been taken up by other (largely non-mathematical) projects.

As I will almost-certainly soon return to a career in finance, it is unlikely I will be able to finish these investigations in the foreseeable future. However I am confident some of these ideas can be made to work (and hopefully to obtain new results) so I thought it would be worth spending a few hours writing this letter.

Needless to say I am sure you already have many ongoing research projects and so you may not have time to think seriously about this material, but I hoped you might know somebody who could be interested. Please feel free to share this correspondence with anybody you think might care to take a look; I would be happy to correspond with any such person and to expand on the below.

As to the ideas I have in mind, my motivation was a desire to connect the ideas in Landsberg's paper [5] with the techniques used in my paper [6]. In [5] Landsberg shows how to obtain the dimension restriction for Severi varieties by using the second fundamental form to construct a certain Clifford module. However ever since Atiyah, Bott and Shapiro's seminal paper [2] it has been known that there are deep connections between Clifford modules and K-

theory. I believe it is worth exploring these connections in the context of secant-defective varieties. In the worst case, one can show how Landsberg's Clifford module structure provides a third proof of the divisibility property for LQEL manifolds (admittedly we probably have enough proofs of this). In the best case it may be possible that the Atiyah-Bott-Shapiro ideas make possible the use of powerful K-theoretic methods for obtaining new results for general secant-defective varieties via Landsberg's Clifford module structure.

Restricting to Severi varieties, what Landsberg essentially[1] shows [5, theorem 6.26] is the following:

Proposition 1. *Let $X \subset \mathbb{P}^N$ be a Severi variety, $Q \subset X$ be a general entry locus, $F \subset Q$ a general tangent locus and $x \in F$ a general point. Then T_xF carries a natural non-degenerate quadratic form and the fibre of the normal bundle $N^x_{Q|X}$ is a Clifford module for the Clifford algebra $Cl(T_xF)$.*

It is not immediately clear when the hypotheses of Landsberg's result hold (though certainly, as stated, for Severi varieties) and which are necessary. Also, he restricts to the case when the secant variety has codimension 1 and I believe this restriction can at least be lifted in the presence of LQEL geometry. Assuming I do not err, proposition 1 in fact holds for any sufficiently-defective LQEL manifold. The divisibility property then follows in view of the following:

Proposition 2. *Suppose we have a complex vector space of dimension n carrying a non-degenerate quadratic form and suppose that we also have an m-dimensional module for the associated Clifford algebra. Then:*

$$2^{[\frac{n}{2}]} \mid m$$

This proposition (which Landsberg seems to have avoided using by means of low-dimensional case analysis) is a trivial consequence of the classification of Clifford modules together with basic facts about representations of matrix algebras. (Interestingly, the dependence on the parity of n thus corresponds to the mod-2 periodicity of Morita equivalence classes of complex Clifford algebras, i.e., it is the same 2 as appears in complex Bott periodicity.) So we have the divisibility property for LQELs since $n = \delta - 1$ and $m = n - \delta$ for the Landsberg Clifford module structure.

Now to make the connection with K-theory, we need to use the ideas of

[1]Unfortunately, although [5] is full of wonderful ideas, it also suffers from an unusually-large number of typos, including the statement of the key Clifford-module structure. Many of these are corrected in the book [4].

Atiyah, Bott and Shapiro (henceforth ABS). ABS establish the following result which is not quite what we need but is a good starting point:

Proposition 3. *Let $V \to X$ be a real vector bundle such that each fibre carries a positive-definite quadratic form and suppose that $W \to X$ is a bundle of graded Clifford modules over the field k ($k = \mathbb{R}$ or $\mathbb{C}$ are both allowed even though V must be real) for the bundle of Clifford algebras $Cl(V)$. Then W determines a natural element of $K_k(Th(V))$ where $Th(V)$ is the Thom space of V and K_k is real or complex K-theory according to k.*

In fact ABS show that this map depends only on the class of W up to a natural equivalence and so the above proposition provides a map that fits into an exact sequence of Abelian K-groups associated to V but this is not relevant here.

Several remarks are in order:

- Proposition 3 deals in terms of *graded* Clifford modules whereas the Landsberg Clifford module is ungraded. This is not a problem as there is a straightforward correspondence between graded and ungraded modules discussed by ABS[2]. For example if V and W are as in proposition 3 except that W is ungraded then $W \oplus W$ is naturally a bundle of graded Clifford modules for the bundle of Clifford algebras associated to the orthogonal direct sum $V \oplus 1$ (with the positive-definite quadratic form on the factor 1).
- Proposition 3 uses *bundles* of Clifford algebras and modules. If we are to apply something like it using the Landsberg Clifford module structure then in the notation of proposition 1 we must let x vary. I believe this is not a problem, although TF or $N_{Q|X}$ may pick up a twisting by $\mathcal{O}(\pm 1)$. In fact I expect it will be $N_{Q|X} \oplus N_{Q|X}(-1)$ that carries the structure of a bundle of graded Clifford module structures. Also since it is $TF(-1)$ that carries a natural quadratic form and since:

  $$TQ(-1) = TF(-1) \oplus \mathcal{O}$$

  is a natural orthogonal decomposition on F, I expect it will be the bundle of graded Clifford modules associated to $TQ(-1)$ that will play a role.
- Proposition 3 applies to *real* Clifford algebras whereas the Landsberg Clifford algebra is complex. This changes the setup in a non-trivial

[2]ABS discuss the case when X is a point but there does not seem to be a problem for general X.

way which I will discuss below.

Regarding the last point mentioned above, in concrete terms, the ABS construction (which yields an element of K-theory from a graded Clifford module) does not work when V is a complex vector bundle. Their construction relies on the property that for a positive-definite real quadratic form q we have $q(v) = 0 \iff v = 0$ for a vector v, which of course is false for any quadratic form in the complex case.

Note also that there is a topological obstruction for a complex vector bundle V to carry a non-degenerate quadratic form: it does so iff $V \simeq V^*$. Furthermore when this holds, by choosing a Hermitian metric we get $V \simeq \bar{V}$ and so V is the complexification of a real vector bundle V_R, say, carrying a positive-definite quadratic form. From this point of view, a complex vector bundle carrying a non-degenerate quadratic form is just the complexification of an underlying real vector bundle with non-degenerate quadratic form *except that we have forgotten the signature of the real quadratic form.*

Related to this then is the ABS construction for non-degenerate indefinite-signature real quadratic forms. Again the naïve ABS construction fails to go through since we no longer have the $q(v) = 0 \iff v = 0$ property but in a subsequent paper [1] to ABS [2], Atiyah shows how to make things work. (He shows how real Clifford modules for real quadratic forms of signature (a, b) give elements of a K-group isomorphic to KO^{a-b} of the base.)

In fact the indefinite-signature real case may be enough to deal with the case of the ABS construction for non-degenerate complex quadratic forms since we can decompose such a form into its real and imaginary components q_R, q_I where are split-signature real quadratic forms, if we forget the complex structure. (This seems plausible since q_R and q_I together determine the forgotten complex structure.)

The issue then is that the combined results of [2] and [1] cover the ABS construction for real and complex modules of definite-signature real quadratic forms as well as real modules of indefinite-signature real quadratic forms. However the case of *complex* modules of indefinite-signature real quadratic forms (which as I say, I think may resolve the case of non-degenerate complex quadratic forms) does not seem to have been discussed anywhere.

Notwithstanding the last four paragraphs, I think I may be making a bit of a meal of this. The complex case should be simpler than the real case! I can almost believe that a result along the following lines might be true:

Not-entirely-serious proposition 4. *Let $W \to X$ be a bundle of graded Clifford modules for the bundle of complex Clifford algebras associated to a complex vector bundle with non-degenerate quadratic form. Then $[W] = \operatorname{rank} W$ in $K(X)$ (i.e., [W] is trivial).*

I do not seriously claim this sort of wild speculation is correct (I have not taken time to think properly about it). It would say that the ABS construction for the complex case is essentially trivial but it would be very useful when applied to the Landsberg Clifford module. For example, it would recover the content of remark 2.7 in my paper [6]. Whatever the truth, I am sure that some it must be possible to establish some simple general result which will link up my work with Landsberg's.

Assuming the above can be done the next thing to do is to try and see if it is possible to move beyond LQEL manifolds into general secant-defective manifolds. As I have mentioned, I find the hypotheses of Landsberg's theorem 6.26 rather opaque and I confess not to have scrutinised his arguments to the level of detail that allows me to see their domain of applicability.

At least since Griffiths and Harris [3] it has been known that global algebro-geometric information is encoded in the local differential-geometry that is the second fundamental form (henceforth SFF). Landsberg makes excellent use of this by (amongst other things) focusing his attention on the Kodaira map associated to the SFF at a point, viewed as a linear system. To use your standard notation (as in [7] for example), given a point $x \in X \subset \mathbb{P}^N$ this is the rational map:

$$\tilde{\pi}_x : \mathbb{P}(T_x X) \dashrightarrow W_x \subset \mathbb{P}(N^x_{X|\mathbb{P}^N})$$

By considering the derivative of this map at an appropriate point we thus have an isomorphism from the normal bundle of (an irreducible component of) a fibre of $\tilde{\pi}_x$ and the tangent space of the corresponding point in the image of $\tilde{\pi}_x$. Fixing x but varying the point in the fibre, Landsberg shows that these isomorphisms fit together into a space of isomorphisms parameterised by $T_x F$ (for a tangent locus F) which he shows satisfy the fundamental Clifford algebra identity. Hence the Clifford module structure.

If Landsberg's construction can be made to work for more general varieties than LQELs (this is far from clear — I have not established how much more than secant-defectiveness he needs to establish the Clifford algebra identity) then, using your notation from [7, definition 2.3.3] I might expect that it is the fibres of the normal bundle of $\Xi = \Xi(X, H)$ (for appropriate H) in X that would carry the interesting algebraic data (e.g., a Clifford module structure?). The dimensions at least look right, especially in view of the refinement of Zak's Linear Normality Theorem that both you [7, equation (3.1.5)] and Landsberg [4, page 135] prove.

Even I find it a bit hard to believe that there could in general be a Clifford module structure on the fibre of the normal bundle $N^x_{\Xi|X}$ for the bundle of Clifford algebras associated to T_xF for a tangent locus F and point $x \in F$ since this would give the extremely-strong-looking divisibility property:

$$2^{[\frac{\delta-1}{2}]} \;\Big|\; n - \xi$$

but as I have said, I believe it must be possible to take these ideas beyond the LQEL setting in some way.

Assuming this could be done, then there would be the task of brining in K-theory, presumably along the lines which I outlined I think will work in the LQEL case. Then finally one hopes there would be a dividend in that we could use K-theory to tackle some open problems.

I apologise for the sketchiness of much of the above, I am trying to tidy up several projects relatively quickly. I admit I have been in two minds as to whether I should even release ideas in such an incomplete form but in the end I decided it could surely do no harm.

Best wishes and happy new year,

Oliver Nash

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